

A Course in Model Theory I:

Introduction¹

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¹This **preliminary draft** is dated from July 27 2020. The book will be published by Cambridge University Press. The book is approximately 99.61% complete, I expect that the final version will have about 800 pages, many sections of the current version will be revised and few will be added. I hope to have a stable version of this volume soon.

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Exercise #=781

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