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Characterizing stability and superstability by unions of chains of saturated models

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Abstract

We show that the non-superstability (non-stability) of a theory T is equivalent to the condition that for every elementary chain of length ω (ω_1) of saturated models M_i of T with $\|M_{i+1}\| > \|M_i\|$ for all $i < \omega\,(\omega_1)$, the union of the chain is not saturated. The cardinality of T is immaterial.

1 Introduction

Our starting point is a result of S.Shelah ([3] III.3.11):

Theorem 1 *If $\{M_i\}_{i<\delta}$ is an increasing sequence of λ-saturated models, and $\kappa(T) \leq cf\delta$ then $M = \cup_{i<\delta} M_i$ is λ-saturated.*

Recall that a theory is superstable iff $\kappa(T) = \aleph_0$ ([3] III.3.8(2)). So for superstable theories the theorem above tells us that the union of any chain of saturated models is saturated. In this paper we prove a converse to this result, and extend it also to unstable theories. Specifically:

Theorem 2 *Let T be a theory. The following are equivalent:*

1. *T is not superstable.*

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2. *There exists an increasing sequence of cardinals* $\{\lambda_n\}_{n<\omega}$, *and an increasing sequence* $\{M_n\}_{n<\omega}$ *of saturated models of* T *with* $\|M_n\| = \lambda_n$ *such that* $M = \cup_{n<\omega} M_n$ *is not* $\aleph_1$*-saturated.*

3. *For every increasing sequence of cardinals* $\{\lambda_n\}_{n<\omega}$, *and every increasing sequence* $\{M_n\}_{n<\omega}$ *of saturated models of* T *with* $\|M_n\| = \lambda_n$ *the model* $M = \cup_{n<\omega} M_n$ *is not* $\aleph_1$*-saturated.*

Furthermore we also have:

Theorem 3 *Let* T *be a theory. The following are equivalent:*

1. T *is not stable.*

2. *There exists an increasing sequence of cardinals* $\{\lambda_i\}_{i<|T|^+}$, *and an increasing sequence* $\{M_i\}_{i<|T|^+}$ *of saturated models of* T *with* $\|M_i\| = \lambda_i$ *such that* $M = \cup_{i<|T|^+} M_i$ *is not* $|T|^{++}$*-saturated.*

3. *For every increasing sequence of cardinals* $\{\lambda_i\}_{i<\omega_1}$, *and every increasing sequence* $\{M_i\}_{i<\omega_1}$ *of saturated models of* T *with* $\|M_i\| = \lambda_i$ *the model* $M = \cup_{i<\omega_1} M_i$ *is not* $\aleph_2$*-saturated.*

In fact we obtain a somewhat sharper result than this. The details will be discussed in the proof, and in the following section.

We use primarily the notation of [3]. In particular if M is a model, then $|M|$ denotes its underlying set, and $\|M\|$ denotes the cardinality of its underlying set. Also, we assume the existence of a "monster model" $\mathfrak{C}$ which is a very large saturated model, containing all the models which we discuss as elementary submodels. We shall refer to "subsets of M" and "elements of M" when we really mean subsets of $|M|^n$ and elements of $|M|^n$ for some fixed finite n.

No intimate knowledge of the details of stability theory is essential. Only we require that if T is not superstable, then a certain tree of elements with special properties exists in $\mathfrak{C}$ (we specify the properties in the proof), and if T is not stable, then some infinite subset of $\mathfrak{C}$ is linearly ordered by a formula $\varphi \in L(T)$.

2 Proofs

In this section we prove Theorems 2 and 3 from above. We begin with the proof of Theorem 2

Proof: First note that (3) implies (2) is trivial, and that (2) implies (1) follows immediately from Theorem 1. So it remains only to prove that (1) implies (3). The assumptions are as follows:

Assumptions: T is a theory which is not superstable; λ is a cardinal of cofinality $\aleph_0$, and $\lambda = \sum_{n<\omega} \lambda_n$ with $\lambda_{n+1} > \lambda_n$ for all $n < \omega$; $\{M_n\}_{n<\omega}$ is an increasing chain of saturated models of T, with $\|M_n\| = \lambda_n$ for all $n < \omega$; and finally $M = \cup_{n<\omega} M_n$.

There is no loss of generality in assuming that $|T| \leq \lambda_0$ since the existence of a saturated model in λ_0 implies that there is a sublanguage of $L(T)$ of cardinality no greater than λ_0, over which all the symbols of T can be defined. This result is due to Keisler [2], or see [3] III.5.14.

We must prove that M is not $\aleph_1$-saturated.

Because T is not superstable, we can find a set:

$$\{\bar{a}_\eta : \eta \in {}^{\omega\geq}\lambda\} \subseteq \mathfrak{C}$$

and formulas:

$$\{\varphi_n(\bar{x}, \bar{y}) \in L(T) : n < \omega\},$$

such that,

$$(\star) \qquad \forall n < \omega\, \forall \eta \in {}^{\omega}\lambda\, \forall \nu \in {}^{n}\lambda \quad \models \varphi_n[\bar{a}_\eta; \bar{a}_\nu] \iff \nu < \eta.$$

(where $<$ is the initial segment ordering on the tree ${}^{\omega\geq}\lambda$).

Let

$$X_n := {}^{n}\lambda_n \ , \ Y_n := \{\bar{a}_\eta : \eta \in X_n\}$$

for all $n < \omega$.

Now form an increasing chain $\{N_n\}_{n<\omega}$ of submodels of $\mathfrak{C}$ such that

(1) $\|N_n\| = \lambda_n$

(2) $N_n \supseteq Y_n$.

This is possible by the downward Löwenheim Skolem Theorem.

Since each M_n is saturated and of cardinality λ_n, there exist elementary embeddings $f_n : N_n \hookrightarrow M_n$ such that $f_n \subseteq f_{n+1}$. Set f to be an automorphism of $\mathfrak{C}$ which extends $\cup_{n<\omega} f_n$.

We will inductively define:

$$\{\eta_n, \nu_n \in X_{n+1} : n \in \omega\}$$

with the following properties:

(1) $\eta_n < \eta_{n+1}$

(2) $\eta_n < \nu_{n+1}$

(3) $T\}_n+i \neq \nu_{n+1}$

(4) $tp(/K_{tt+1}), M_n) = tp(f(\bar{a}_{\nu_{n+1}}), M_n)$.

To do this, set 70 = 0, anc^ then suppose that *{rjk, v* : k ≤ n}* are defined and satisfy (l)-(4). Since M_{n+1} is saturated and $||M_n+i|| > ||M_n||$, every type over $\backslash M_n\backslash$ is realized in $M_{n+}i$. So in particular, $|5(M_n)| \leq A_{n+}i$. But since

$$\backslash X_n+_2\backslash = A_{n+2} > \backslash S(M_n)\backslash$$

there are two distinct sequences $i/_n+i$ > *7n+i € -Xn+2 both extending $r;_n$ and with $tp(/(\bar{a},_{n+1}), M_n) = tp(/(\bar{a},_{w+1}), M_n)$.

Having defined these sequences, let

This type p is consistent since it is realized by /(a^) where 77 := $U_n eu > Vn$ (this follows from (*). However, *p* is omitted by *M.* For if $\bar{a}$ G *M* then $\bar{a}$ G M_n for some **nGw.** Thus by (4):

$$\models \varphi_{n+2}[\bar{a}; f(\bar{a}_{\eta_{n+1}})] \iff \varphi_{n+2}[\bar{a}; f(\bar{a}_{\nu_{n+1}})]$$

and so $\bar{a}$ does not realize *p.* *m* (of Theorem 2)

Now for the proof of Theorem 3:

Proof: Again (3) implies (2) is trivial. (2) implies (1) follows from Theorem 1 since, if *T* is stable, then $K(T) \leq |T|^+$. For (1) implies (3) we will prove something much stronger, namely:

> If *T* is not stable then V/c ≥ Ni, and every increasing sequence of cardinals $\{A»\}_{i</C}$, and every increasing sequence of saturated models $\{Mi\}_{i<K}$ such that ||M,|| = A,, the model *M* = U,<«M, is not $/c^+$-saturated.

First we need a small lemma:

Lemma 4 *If N is a saturated model of an unstable theory T} <p(x;y)* G *L(T) and U is an infinite subset ofN, linearly ordered by <p, and maximal with respect to this property, then U has a final segment B, with \B\ ≤* K_o, *such that U — B is infinite and has no maximal element.*

Proof: If *U* has no maximal element, then *B* = 0 works. Likewise, if there is a finite final segment *B* such that *U* — *B* has no maximal element then there is nothing to prove. Otherwise, there is a final segment *B* of *U* which is isomorphic to a;*. For any $\bar{a}$ G *U* — *B}* the type

$$\{\varphi(\bar{a}; \bar{x})\} \cup \{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B\}$$

is realized in N, hence, due to the maximality of U, $U - B$ can have no maximal element (the same type without the first set shows that $U - B \neq \emptyset$).

■ (of Lemma 4)

Now let λ_i, M_i, and M be as above. Since T is not stable, we can find an infinite subset U_0 of M_0 which is linearly ordered by a formula in $L(T)$. Take $B \subseteq U_0$ as provided by Lemma 4. We shall inductively construct sets $U_i \subseteq M_i$, and elements $\bar{a}_{i+1} \in M_{i+1}$ as follows. We set U_i to be any subset of M_i containing $\cup_{j<i} U_j$ which is linearly ordered by φ, and is maximal with respect to this property. Then B is still a final segment of U_i, and $U_i - B$ has no maximal element. This is because from the maximality of U_0 we may conclude that it was inconsistent to add any element above any element of B, and, as in the proof of Lemma 4, $U_i - B$ can never have a maximal element. Now we construct a sequence $\bar{a}_{i+1} \in U_{i+1}$ by choosing $\bar{a}_{i+1}$ to realize the type:

$$\{\varphi(\bar{a};\bar{x}) : \bar{a} \in U_i\} \cup \{\varphi(\bar{x};\bar{b}) : \bar{b} \in B\}$$

(this is possible since $\lambda_i < \lambda_{i+1}$). Finally, we see that the type:

$$\{\varphi(\bar{a}_{i+1};\bar{x}) : i < \kappa\} \cup \{\varphi(\bar{x};\bar{b}) : \bar{b} \in B\}$$

is consistent but not realized in M, hence M is not κ^+-saturated.

■ (of Theorem 3)

3 Discussion

Here we will make some observations which may clarify or illuminate the theorems above. It is our contention that these theorems provide an interesting and surprising characterization of stability and superstability. The observations below are meant to illustrate this.

Corollary 5 *A theory T is superstable iff every special model of T is saturated. A theory T is stable iff every special model of T whose cardinality is of uncountable cofinality is saturated. Moreover, each of these equivalences remains true if "every" is replaced by "some".*

Also, we should note that we were unnecessarily restrictive in our choice of cardinalities. Inspection of the proof will reveal that in Theorem 2 we could have assumed, instead of "M_n is saturated, and $\|M_n\| = \lambda_n$", only that:

(1) M_n is λ_n-saturated, and

(2) $\sum_{n<\omega} \|M_n\| = \sum_{n<\omega} \lambda_n$.

Similarly, in Theorem 3, instead of "M_i is saturated, and $\|M_i\| = \lambda_i$",

(1) M_i is λ_i-saturated, and
(2) $\sum_{i<\kappa} \|M_i\| = \sum_{i<\kappa} \lambda_i$,

is sufficient.

Note, however that $\lambda = \sum_{i<cf\,\lambda} \lambda_i$ is necessarily a strong limit cardinal, since the existence of a saturated model in λ_i implies $\lambda_i^{<\lambda_i} = \lambda_i$ (by [3] VIII.4.7). So in this case the first cardinal in which we obtain an interesting example is $\beth_{\omega_1}$. However, in Theorem 2 when $\lambda = \sum_{n<\omega} \lambda_n$ then λ need not be a strong limit cardinal, and in fact may be fairly small. For example, if $|L(T)| = \aleph_0$ and $2^{\aleph_0} < \aleph_\omega$ then we have an example of a non $\aleph_1$-saturated special model of cardinality $\aleph_\omega$. We can also strengthen the Theorem 3 somewhat, in the equivalence of (1) and (2), since the requirement that the cardinalities be increasing is not necessary. Specifically:

Proposition 6 *Let T be a theory. The following are equivalent:*

1. *T is not stable.*
2. *For any λ and any κ, with $\aleph_0 \leq \kappa < \lambda$, if T has a saturated model M with $\|M\| = \lambda$, then there exists an increasing chain of models $\{M_i\}_{i<\kappa}$ such that $\forall i < \kappa\, M_i \cong M$ but $\cup_{i<\kappa} M_i \not\cong M$ (in fact is not κ^+-saturated).*

Proof: This result is a slight generalization of one in [1]. But the proof is very easily placed in the context of the proof of Theorem 3. The chain $\{M_i\}_{i<\kappa}$ is constructed by beginning with $M_0 \cong M$, and with U_0 and B as in the proof of Theorem 3. Now, it is consistent to add elements between $U_0 - B$ and B and, as M is a universal model, we can construct M_1 so that it contains such elements. The rest of the construction proceeds in this fashion, and the proof exactly as above. ∎

Also, the theorems give us a characterization of stable, or superstable theories T in terms of the category $Mod(T)$ of models of T and elementary embeddings, and the full subcategory $Sat(T)$ of saturated models of T.

Corollary 7 *T is superstable iff $Sat(T)$ is closed in $Mod(T)$ under direct limits. T is stable iff $Sat(T)$ is closed in $Mod(T)$ under direct limits of uncountable cofinality.*

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