Advances in Classification Theory for Abstract Elementary Classes

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"If I have seen further it is by standing on the shoulders of giants." – Isaac Newton

While I would never claim to have seen as far as Isaac Newton, the extra distance I have seen is a direct result of others who, more than just letting me stand on their shoulders, have actively helped me up to admire the view.

My parents always encouraged me to excellence and taught me more than I could ever acknowledge, from my youth as an obnoxious and difficult child to this day, when I am a (hopefully) less obnoxious and difficult adult.

My teachers and mentors over the years have had a similar impact in educating and (perhaps more importantly) encouraging me on the long path from addition to this thesis. I am grateful to all of them for their hard work and commitment to the greatest profession.

In a different direction, I am grateful to my friends, in Pittsburgh and beyond. While a local analysis might conclude that they delayed this thesis, more global considerations reveal that their distractions were a net positive.

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Finally, I'd like to dedicate this thesis to my wife, Emily Boney. Her encouragement and support were invaluable in the completion of my research. Beyond that, she is a constant source of joy and happiness for me. I love you.

In this thesis, we continue the project of classification theory for Abstract Elementary Classes (AECs), especially tame AECs. Chapter I contains a general introduction and Chapter II provides preliminaries.

Chapter III, "Types of Infinite Tuples," analyzes Galois type by their length. We show that the number of types of sequences of tuples of a fixed length can be calculated from the number of 1-types and the length of the sequences. Specifically, if $\kappa \leq \lambda$, then

$$\sup_{\|M\|=\lambda} |S^\kappa(M)| = \left(\sup_{\|M\|=\lambda} |S^1(M)| \right)^\kappa$$

We show that this holds for any abstract elementary class with λ amalgamation. Basic examples show that no such calculation is possible for nonalgebraic types. However, we introduce a generalization of nonalgebraic types for which the same upper bound holds.

Chapter IV, "Tameness and Large Cardinals," uses large cardinals to derive locality results for AECs. The main success is showing that Shelah's Eventual Categoricity Conjecture for Successors follows from the existence of class many strongly compact cardinals. This is the first time the consistency of this conjecture has been proven. We do this by showing that every AEC with $LS(K)$ below a strongly compact cardinal κ is $< \kappa$ tame and applying the categoricity transfer of Grossberg and VanDieren. We obtain similar, but weaker results, from measurable and weakly compact cardinals. We introduce a dual property to tameness, called *type shortness*, and show that it follows similarly from large cardinals.

Chapter V, "Nonforking in Short and Tame Abstract Elementary Classes," uses the conclusions of the previous chapter to develop a notion of forking for Galois-types in the context of AECs. Under the hypotheses that an AEC K is tame, is type-short, and fails an order-property, we consider

Definition. *Let $M_0 \prec N$ be models from K and A be a set. We say that the Galois-type of A over M does not fork over M_0, written $A \underset{M_0}{\downarrow} N$, iff for all small $a \in A$ and all small $N^- \prec N$, we have that Galois-type of a over N^- is realized in M_0.*

Assuming property (E), we show that this non-forking is a well-behaved notion of independence. In particular, it satisfies symmetry and uniqueness and has a corresponding U-rank. We find sufficient conditions for a universal local character and derive superstability-like property from little more than categoricity in a "big cardinal." Finally, we show that under large cardinal axioms the proofs are simpler and the non-forking is more powerful.

Chapter VI, "Tameness and Frames," combines tameness and Shelah's good λ-frames. This combination gives a very well-behaved nonforking notion in all cardinalities. This helps to fill a longstanding gap in classification theory of tame AECs and increases the applicability of frames. Along the way, we prove a complete stability transfer theorem and uniqueness of limit models in these AECs.

Chapter VII, "A Representation Theorem for Continuous Logic," details a correspondence between first-order continuous logic and $L_{\omega_1,\omega}$. In particular, for every continuous object (language, structure, etc.), there is a discrete analogue. This discrete analogue requires an infinitary description to ensure the range of the (analogue of the) metric has range in the real numbers. This correspondence can be inverted and we extend it to types and saturation.

Chapter VIII, "A New Kind of Ultraproduct," explores a tension revealed in Chapter VII: first-order continuous logic is compact, but $L_{\omega_1,\omega}$ is, in general, not. The explanation for this tension is the Banach space ultraproduct. This chapter develops a general model-theoretic construction $\Pi^\Gamma M_i/U$ that attempts to capture the properties of the Banach space ultraproduct.

Chapter IX, "Some Model Theory of Classically Valued Fields," applies some ideas from classification theory to a specific AEC: the class of classically valued fields. The main tool is the analytic ultraproduct, but its development is entirely self-contained. The classic version of Łoś' Theorem fails for this ultraproduct, but an approximate version is proved.

Chapter 1

Introduction

Model theory is a branch of mathematical logic that seeks to strip away the nonessential structure of mathematical objects so that similarities between them can be recognized. It is, in some sense, a descendant of abstract algebra which found the group as a common structure in the addition of numbers, the composition of functions, and the automorphisms of a field. Where algebra replaces these with a function (the group operation) and a list of axioms (the group axioms), model theory looks at structures in some arbitrary language (a collection of functions and relations) that model some arbitrary theory (a set of axioms in that language). The bulk of model theory works in the context of first order theories, which allow finite quantification of elements and finite boolean combinations of formulas.

The classification of first order-theories has been one of model theory's recent successes. This field began in 1962 with Morley's Categoricity Theorem from Morley's thesis, see [Mor65] for the published version.

Theorem 1.0.1 (Morley). *If T is a theory in a countable language that is categorical in some uncountable cardinal, then it is categorical in every uncountable cardinal*

This theorem (and Shelah's later generalization of it to uncountable theories in [Sh31]) state that, given a theory, if there is only one model of it (up to isomorphism) in some big cardinal, then there is only one model of it (up to isomorhpism) in *every* big cardinal; here, "big" is taken to mean "larger than the theory." This allows one to conclude, for instance, that since there is only one algebraically closed field with characteristic zero of size $\aleph_1$, then there is only one algebraically closed field of characteristic zero of any uncountable size.

Coarsely viewed, Morley's theorem implies that countable first-order theories are either very well behaved (categorical in *every* uncountable cardinal) or very poorly behaved (categorical in *no* uncountable cardinal). This gap between well behaved and poorly behaved, with no example of a theory in between, is typical of classification theory. Stronger than this is the notion of a dividing line. In addition to having this gap, a dividing line will imply interesting theorems on both sides (a theory not being categorical does not seem to imply anything interesting).

Classically, the main dividing line has been between stable and unstable theories. This contrasts well behaved theories–those that have few types–with poorly behaved theories–those that have an infinite, definable linear order. Notice here another key feature of dividing lines: both sides are characterized by positive properties, rather than just being the negation of the other.[1] That there are no theories in between is a consequence of the Unstable Formula Theorem.

Theorem 1.0.2 ([Sh:c].II.2.2, Partial). *Fix a theory T. The following properties of a formula ϕ are equivalent:*

1. *For every $\lambda \geq \aleph_0$, there is A of size λ such that $|S_\phi(A)| > \lambda$.*

2. *There is some $\lambda \geq \aleph_0$ such that there is A of size λ such that $|S_\phi(A)| > \lambda$.*

3. *ϕ has the order property.*

Initially, stable and stronger (superstable, NDOP, NOTOP, etc.) theories dominated the landscape of classification theory, although many unstable dividing lines (including the ones below) were introduced by Shelah in [Sh:a]. However, in the last fifteen years, the classification of unstable theories has been

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[1]Of course, this negation characterization also holds.

advanced by work on simple theories (Kim [Ki98] and Kim and Pillay [KP97]), NIP theories (see surveys by Adler [Ad09] and Simon [Si]), and, most recently, NTP_2 (Ben-Yaacov and Chernikov [BYCh] and Chernikov, Kaplan and Shelah [CKS1007]). The standard reference for this subject is Shelah's aptly named book **Classification Theory and the number of nonisomorphic models** [Sh:c] and an excellent visual map of the current dividing lines has been created by Conant [Con] online at `http://www.forkinganddividing.com/`.

One of the main tools of classification theory for first-order theories is nonforking, first identified by Shelah in [Sh:a]. Sometimes called an independence notion, nonforking generalizes the notion of linear dependence to arbitrary, well behaved, first-order theories. Different dividing lines correspond to different strengths of nonforking, and where a theory falls in these dividing lines can often be identified by the particular properties that nonforking has in that theory.

As far reaching as this project has been, it is limited by the restrictions of first-order logic. While first-order logic is powerful, the compactness theorem implies that many mathematically interesting properties are not axiomatizeable: any infinite torsion group is elementarily equivalent to a group with an element of infinite order; the natural numbers are elementarily equivalent to nonstandard models of Peano Arithmetic; the reals (as an ordered abelian group) are elementarily equivalent to non-Archimedean ordered groups, etc. In a sense, the compactness theorem can be seen as a strength and a weakness of first-order logic: it can be used to realize any finitely consistent type, but it also means that any set of sentences that doesn't contradict a finite part of a theory must hold in some model of that theory.

Beyond first-order logic, there are a host of stronger logics to choose from. $L_{\omega_1,\omega}$ is the best studied of the infinitary logics, but is already strong enough to capture each of the properties described in the previous paragraph; Keisler's book [Kei71] is a good reference. Stronger infinitary logics allow for the conjunction of more formulas and for the quantification of more variables, from $L_{\lambda,\kappa}$ up to $L_{\infty,\infty}$; Dickman [Dic75] is a good reference here. Other logics add extra quantifiers, such as the cardinality quantifier Q_λ, which was introduced by Mostowski [Mos57], or the Ramsey-like quantifiers of Magidor and Malitz [MM77].

In 1977, Saharon Shelah circulated the paper that would eventually become [Sh88].[2] This paper contained the definition of an Abstract Elementary Class (see Definition 2.1.1; all relevant definitions are given in the next chapter). An Abstract Elementary Class, or AEC, is a collection of models in a fixed language, along with a strong substructure relation that satisfies a set of axioms. These AEC axioms reflect basic model-theoretic facts that can be proved about models of a first-order theory *without* the compactness theorem. This avoidance of the compactness theorem means that this framework includes most of the logics discussed above as special cases.[3]

A natural outgrowth of this work is the classification theory of Abstract Elementary Classes. As suggested by the title, this thesis focuses on this subject. Classification theory for AECs seeks to find dividing lines, similar to the first-order case, but also to resolve issues that don't exist in elementary classes. These goals are exemplified by the two main test questions in the field:

1. Shelah's Categoricity Conjecture is the analogue of Morely's Theorem for AECs. An early version appeared in the list of open problems from [Sh:c] as D.(3a), but is often stated in the following way.

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[2]This would later be revised to [Sh88r], the first chapter of his two-volume work **Classification Theory for Abstract Elementary Classes** [Sh:h].

[3]The exception to this are the logics with quantification for infinite tuples. L_{ω_1,ω_1} is already strong enough to express well-ordering; thus, many basic questions about this logic turn into set-theoretic questions.

Conjecture 1.0.3 (Shelah). *For every λ, there is some μ_λ such that if K is an AEC with $LS(K) = \lambda$ that is categorical in a cardinal greater than or equal to μ_λ, then it is categorical in every cardinal greater than or equal to μ_λ.*

Note that "categorical above the size of the language" in the first-order version has been replaced by "categorical above some threshold." This speaks to its openness and it is unresolved even for concrete AECs, such as those axiomatized by $L_{\omega_1,\omega}$. Moreover, most results on this conjecture further assume the categoricity cardinal is a successor. Major milestones in work on this conjecture are Shelah [Sh394] (downward transfer from a big cardinal for classes with amalgamation); Makkai and Shelah [MaSh285] (for AECs axiomatized by $L_{\kappa,\omega}$ when κ is strongly compact); Kolman and Shelah [KoSh362] and Shelah [Sh472] (weaker results than those of Makkai and Shelah but just using κ measurable); Grossberg and VanDieren [GV06c] and [GV06a] (upward transfer for tame AECs with a monster model); and Hyttinen and Kesälä [HyKe06] [HK07] and [HK11] (for finitary AECs, but without any successor assumptions).

2. The other main test question is about the existence of models at some cardinal, given some strong assumptions about the class below. Compactness gives an easy answer to this question in the first-order case, but it is easy to create $L_{\omega_1,\omega}$ sentences with maximal models; the classic example is to axiomatize substructures $(V_\alpha, \in)$ for fixed countable α. The strongest version of this question can be phrased as follows.

Question 1.0.4. *Let K be an AEC and $\lambda \geq LS(K)$. If K is λ- and λ^+-categorical, does K have a model of size λ^{++}?*

Above the Hanf number of a class, the existence of arbitrarily large models follows, so this question is normally reserved for when $\lambda < \beth_{(2^{LS(K)})^+}$. The first major work on this question was Shelah [Sh576], which added the assumption that $1 \leq I(\lambda^{++}, K) < \mu_{wd}(\lambda^{++})$ and $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ to get a model in λ^{+3}. This work further spawned good λ-frames (described below) in Shelah [Sh600], which is a nonforking relation on λ-sized models that restricts its attention to a distinguished set of nonalgebraic types. The goal of this project is to inductively build up structure theory for an AEC from its local behavior; some work on this project attempts to answer both of these test questions uniformly.

There are also classification questions that span the two directions described above, such as the uniqueness of limit models. Limit models, previously called brimmed or (λ, α)-saturated models, are models that are the union of a universal chain of models (i.e. M_{i+1} is universal over M_i) of a fixed cardinality. If these chains start with the same base model, then any two limit models whose chain length are of the same cofinality are isomorphic over that base by a back and forth argument. If this is true for chains of any length, then the AEC is said to have unique limit models. This has become a well-studied question and is a candidate for a dividing line in AECs that is not seen in first-order; see Kolman and Shelah [KoSh362]; Shelah [Sh600]; Shelah and Villaveces [ShVi635]; VanDieren [Van06] [Van13]; and Grossberg, VanDieren, and Villaveces [GVV].

This thesis is thematically split into two parts (excluding this introduction and the preliminary chapter): the first part focuses on more general results in classification theory and consists of Chapters III, IV, V, and

VI. The second part considers more concrete examples of AECs and applies some of these classification results to them; this consists of Chapters VII, VIII, and IX. The rest of the introduction gives some introduction and motivation for this work, but assumes more familiarity with Abstract Elementary Classes. An option for unfamiliar readers is to peruse the next chapter for definitions and then come back here for motivation.

One of the key innovations in AECs has been the introduction of Galois types by Shelah in [Sh300]. Just as syntactic types have become ubiquitous in first-order model theory, so have Galois types become used throughout work in classification theory for AECs since their introduction. It is not hard to see that first-order types (as sets of formulas) are not useful in AECs: the class might be "stronger" (e.g. axiomatized by $L_{\omega_1,\omega}$) or "weaker" (e.g. abelian groups with the subgroup relation) than first order and, thus, lose the tight connection with first-order logic that elementary classes have. Even if the AEC comes equipped with a logic, which is not always the case, the power of first-order types comes from some of the special features of first-order logic, such as compactness and interpolation, and does not transfer to types in other logics.

Instead, Galois types take some of the key semantic consequences of syntactic types and make this the definition of two types being equal. This definition has turned out to be a great tool in the analysis of AECs. Certain definitions based on types in first-order–stability through counting types, saturation, etc.–have been imported to AECs with great success. Much of the analysis has concentrated on types of single elements or of finite tuples. However, some of the analysis in this thesis (and before: Makkai and Shelah [MaSh285]; Shelah [Sh:h].V; and Grossberg and VanDieren [GV06b] are examples) relies on types of infinite tuples. In exploring and defining these types, we discover new results on counting the number of types of infinite tuples; this appears in Chapter III.

Theorem 1.0.5. *If K is an AEC with λ-amalgamation and $\kappa \leq \lambda$, then*

$$\sup_{M \in K_\lambda} |gS^\kappa(M)| = \left(\sup_{M \in K_\lambda} |gS^1(M)| \right)^\kappa$$

Unfortunately but expectedly, Galois types are not as well behaved as their first-order cousins. Concretely, this can be seen in the examples constructed by Hart and Shelah [HaSh323] (revisited with this issue in mind by Baldwin and Kolesnikov [BK09]) and by Baldwin and Shelah [BlSh862].[4] In both of these examples, there are types that are "wild" in the following sense: the type is not determined by its smaller restrictions. Equivalently, there are two types that look the same when restricting to every smaller type, but are in fact different. This is easily contrasted with first-order types, where any difference of two types is witnessed by a formula, which contains only finitely many parameters and free variables. In the Hart and Shelah examples, Baldwin and Kolesnikov identified that this wildness happens for types with domain of size $\aleph_k$ for any fixed $k < \omega$. In the Baldwin and Shelah example, this wildness occurs at κ whenever there is an almost-free, non-Whitehead group of size κ. We discuss the specifics of this last case more at the end of Chapter IV, but we mention now that this happens at $\kappa = \aleph_1$ as a result of ZFC and is dependent on set theory for larger κ.

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[4]It should be noted that the order of our description does not align with the historical order of development. Hart and Shelah's work came first, but they studied categoricity rather than the behavior of types. It wasn't until Grossberg and VanDieren defined tameness (introduced below) and proved their categoricity theorem that Baldwin and Kolesnikov looked at the types of the Hart and Shelah example and that Baldwin and Shelah developed their example.

To overcome this wild behavior, Grossberg and VanDieren introduced *tameness* in [GV06b], which came from the latter's thesis [Van02]. A weaker notion, now called *weak tameness*, had been used by Shelah in the midst of his work in [Sh394]. Tameness essentially says that the wildness described above–in particular, with relation to the domain of the type cannot occur. A dual notion, *type shortness*, which states that a similar wildness with respect to the length of the type cannot occur, is by the author in [Bona], which is Chapter IV of this thesis. Both notions are formally defined in the next chapter.

These locality notions are powerful in part because they allow the import of even more techniques and intuition from first-order model theory. Stability transfer results by Baldwin, Kueker, and VanDieren [BKV06]; Morley rank by Lieberman [Lie13]; and more have been developed for tame AECs. Perhaps most strikingly, Grossberg and VanDieren in [GV06c] and [GV06a] have given an affirmative answer to Shelah's Categoricity Conjecture for tame AECs under some standard structural assumptions, i.e. the existence of a monster model.

Theorem 1.0.6 ([GV06a])**.** *Suppose K is an AEC with amalgamation, joint embedding, and no maximal models. If K is χ-tame and λ^+-categorical for some $\lambda \geq LS(K)^+ + \chi$, then K is μ-categorical in all $\mu \geq \lambda$.*

This leads to the natural question of which AECs are tame. Of course, the wild AECs above of Hart and Shelah and of Baldwin and Shelah are not tame. On the other hand, Grossberg and VanDieren already observed in [GV06a] that all known examples of AECs that have a well-developed independence relation turn out to be tame; this observation is made precise in Chapter V. To achieve a more global result, we turn to large cardinals in Chapter IV. Large cardinals have previously seen use in AECs axiomatized by $L_{\kappa,\omega}$, in particular, by Makkai and Shelah [MaSh285] with κ strongly compact and by Kolman and Shelah [KoSh362] and Shelah [Sh472] with κ measurable. Chapter IV extends these results to general AECs and proves global tameness properties from the existence of strongly compact, measurable, or weakly compact cardinals. The main result along these lines is the following.

Theorem 1.0.7 (4.2.5)**.** *If K is an AEC and κ is strongly compact such that $LS(K) < \kappa$ strongly compact, then K is $< \kappa$-tame.*

This allows us to prove the consistency of Shelah's Eventual Categoricity Conjecture for Successors from large cardinals in Chapter IV. We add "Eventual" to denote that the threshold is not computable from the class. Most other works consider a threshold bounded by some low iteration of the Hanf number.

Theorem 1.0.8 (4.5.5)**.** *If there are class many strongly compact cardinals, then Shelah's Eventual Categoricity Conjecture for Successors is true.*

Beyond focus on these test questions, much work has been done in exploring notions of nonforking for AECs. Obviously, the first-order version of nonforking will not work for AECs, so a new definition is needed. Some of this work has focused on versions of splitting and strong splitting introduced by Shelah in [Sh394]. From first-order, we expect that non-splitting is not exactly the right independence notion. However, this has still seen productive use in the work of Grossberg, Shelah, VanDieren, Villaveces, and others.

In Chapter V, which is joint work with the author's adviser Rami Grossberg, we introduce a nonforking notion that generalizes nonforking in stable first-order theories to AECs that are tame and type short. In

particular, we define $\perp$ to be the coheir or $< \kappa$-satisfiability relation. Under reasonable assumptions, this behaves as desired.

Theorem (5.3.1). *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If there is some $\kappa > LS(K)$ such that*

1. *K is fully $< \kappa$-tame;*

2. *K is fully $< \kappa$-type short;*

3. *K doesn't have an order property; and*

4. *$\perp$ satisfies existence and extension,*

then $\perp$ is an independence relation.

We go on to argue that this nonforking relation is robust: we give conditions for when $\perp$ is equivalent to the generalization of heir; we use $\perp$ to give a sufficient conditions for the uniqueness of limit models; and we develop the U-rank corresponding to $\perp$. Other work with Grossberg, Kolesnikov, and Vasey in [BGKV] furthers this argument by proving, under the same hypothesis, that this nonforking is the only one.

Another independence notion that has been heavily explored is Shelah's notion of good λ-frames. Drawing on results regarding the existence test question in Shelah [Sh576], Shelah in [Sh600] defined good λ-frames by providing a list of axioms for a nonforking relation and a set of distinguished types (called *basic*) that the nonforking operates on. A key feature of good λ-frames is that they only deal with models of size λ. The project of good λ-frames–explored in Shelah [Sh600], [Sh705], [Sh734]; Jarden and Shelah [JrSh875]; Jarden and Sitton [JrSi13]; and elsewhere–is to inductively build up frames of larger and larger cardinality. This complex process proceeds, in part, by shrinking the AEC and changing the strong substructure relation and relies heavily on set-theoretic assumptions, such as instances of the weak continuum hypothesis and the non-saturation of certain weak diamond ideals. In Chapter VI, we greatly simplify this process by substituting tameness and amalgamation for the other factors.

In the second, more concrete part of this thesis, we look at specific examples of AECs (or near AECs) and apply the ideas from classification theory to them. Continuous logic has a long history dating back to the work of Chang and Keisler [CK66] and the many-valued logics of Łukasiewicz. A good reference for the modern presentation is Ben Yaacov, Berenstein, Henson, and Usvyatsov [BBHU08]. Continuous logic provides a framework for complete analytic structures: metric spaces, Banach spaces, etc. Recent work has achieved success in part because of collaboration between logicians and analysts. Due to its reliance on complete structures, an L_{ω_1, ω_1} property, and its use of $[0, 1]$-valued formulas, continuous logic is outside the scope of Abstract Elementary Classes. However, in Chapter VII, we give a presentation theorem in the style of Chang's Representation Theorem that shows that the topological completeness of a continuous structure is the only real barrier to doing this analysis inside of an AEC. We do this by capturing the structure of an elementary class of continuous structures by axiomatizing the class of their dense subsets with an $L_{\omega_1, \omega}$ theory. We explore this correspondence and eventually characterize AEC concepts, such as Galois types, in this class of dense sets.

The final two chapters revolve around variants of the ultraproduct construction. In Chapter IV and the end of Chapter V, we used sufficiently complete ultrafilters to prove strong results about AECs. However,

the specific details of the ultraproduct construction were not crucial; all that was needed was a way to average structures of the AEC into another structure of the AEC. An alternate way to do this is a variant of the standard model-theoretic ultraproduct that works well for metric spaces and similar contexts. This variant was introduced by Dacunha-Castelle and Krivine [DCK72] and is a key tool in continuous first-order logic because it proves the continuous compactness theorem. In Chapter IX, we apply this ultraproduct to the class of classically valued fields. Classically valued fields are fields that have value group a subset of the reals. We begin the project of developing the model theory of such spaces by proving an Approximate Łoś' Theorem.

Chapter VIII provides a further variant on the Banach space ultraproduct. From a model-theoretic perspective, this ultraproduct does two novel things: it avoids elements of infinite norm by excluding them and avoids elements of infinitesimal norm by making them equal to 0. While the second step relies on the specifics of the metric space structure, the first step can be viewed as omitting a type simply by excluding any element that would realize it. We generalize this to an arbitrary framework of structures omitting a type by introducing $\Pi^\Gamma M_i/U$. Under ideal circumstances, this is a structure and satisfies Łoś' Theorem; this would allow us to apply the classification results from Chapters IV and V. However, circumstances are often not ideal, so we provide some examples and sufficient conditions for an ideal world.

Chapter 2

Preliminaries

In this chapter, we give the basic definitions used throughout this thesis. It is important to note that we have not made an effort to give an accurate representation of the history or motivation of the subjects, although this sometimes slipped out. Instead, we have focused on a more logical (to the author) presentation of mathematical concepts. For a better discussion of the motivation and history, see the previous chapter or the standard references.

The standard references for this material are books by Baldwin [Bal09], Shelah [Sh:h], and Grossberg [Gro1X] and the survey article by Grossberg [Gro02].

2.1 Abstract Elementary Classes

The central object in this thesis is an Abstract Elementary Class.

Definition 2.1.1. *We say that $(K, \prec_K)$ is an Abstract Elementary Class (AEC) iff*

1. *There is some language $L = L(K)$ such that every element of K is an $L(K)$-structure;*

2. *$\prec_K$ is a partial order on K;*

3. *for every $M, N \in K$, if $M \prec_K N$, then $M \subseteq_L N$;*

4. *$(K, \prec_K)$ respects $L(K)$ isomorphisms, if $f : N \to N'$ is an $L(K)$ isomorphism and $N \in K$, then $N' \in K$ and if we also have $M \in K$ with $M \prec_K N$, then $f(M) \in K$ and $f(M) \prec_K N'$;*

5. *(Coherence) if $M_0, M_1, M_2 \in K$ with $M_0 \prec_K M_2$; $M_1 \prec_K M_2$; and $M_0 \subseteq M_1$, then $M_0 \prec M_1$;*

6. *(Tarski-Vaught chain axioms) suppose $\langle M_i \in K : i < \alpha \rangle$ is a $\prec_K$-increasing continuous chain, then*

   a) *$\cup_{i<\alpha} M_i \in K$ and, for all $i < \alpha$, we have $M_i \prec_K \cup_{i<\alpha} M_i$; and*

   b) *if there is some $N \in K$ such that, for all $i < \alpha$, we have $M_i \prec_K N$, then we also have $\cup_{i<\alpha} M_i \prec_K N$; and*

7. *(Löwenheim-Skolem number) There is an infinite cardinal $\lambda \geq |L(K)|$ such that for any $M \in K$ and $A \subset M$, there is some $N \prec_K M$ such that $A \subset |N|$ and $\|N\| \leq |A| + \lambda$. We denote the minimum such cardinal by $LS(K)$.*

As the name suggests, AECs are an abstraction of elementary classes $(\text{Mod } T, \prec)$, where T is a first-order theory and $\prec$ is the elementary substructure relation for $L(T)$. The axioms listed above are some of the basic properties of elementary classes that can be proved without the compactness theorem. The avoidance of compactness is key as it allows AECs to encompass many non-compact frameworks. The following examples (and more) can be found in Baldwin [Bal09].§4 and [Bal07].

- Infinitary Logic
  Let T be an $L_{\lambda,\omega}$ theory and $\mathcal{F} \subset L_{\lambda,\omega}$ be a fragment containing T. Then $(\text{Mod } T, \prec_\mathcal{F})$ is an AEC with Löwenheim-Skolem number $|\mathcal{F}|$, where $\prec_\mathcal{F}$ is $\mathcal{F}$-elementary substructure.

- Cardinality Quantifiers

  Let Q_λ be the quantifier where "$Q_\lambda x\phi(x)$" means "there are at least λ-many a such that $\phi(a)$ holds." Then $L(Q_\lambda)$ is the logic formed by adding Q_λ to first-order with the obvious syntax and semantics. Let T be a theory in $L(Q_\lambda)$. Then $(\text{Mod } T, \prec^*)$ is an AEC with Löwenheim-Skolem number λ, where $M \prec^* N$ iff M is an $L(Q_\lambda)$-elementary substructure of N and

  $$N \vDash \neg Q_\lambda x\phi(x; \mathbf{m}) \implies \phi(M, \mathbf{m}) = \phi(N, \mathbf{m})$$

- $\forall\exists$-theories

  Let T be an $\forall\exists$-theory, such as the theory of groups or graphs. Then $(\text{Mod } T, \subset)$ is an AEC with Löwenheim-Skolem number $|T|$.

- Classification over a predicate

  Let T be a theory that makes reference to a specific structure N; in practice, this is often a substructure of an expansion of the reals. We want to consider the class of models of T along with the structure N. This is likely not first-order axiomatizable and is also not an AEC because it is not closed under isomorphisms. Instead, set

  $$
  \begin{aligned}
  L^* &= L \cup L(N) \cup \{c_n : n \in N\} \cup \{S(\cdot)\} \\
  T^* &= T \cup ED(N) \cup \{S(c_n); n \in N\} \cup \{\forall x(S(x) \to \vee_{n \in N} x = c_n)\} \\
  F^* &= \text{the smallest fragment of } L_{\|N\|^+, \omega} \text{ that contains } T^*
  \end{aligned}
  $$

  Then $(\text{Mod } T, \prec_F)$ is an AEC with Löwenheim-Skolem number $\|N\| + |T|$ such that each model comes equipped with an isomorphism to a structure from the intended class.

Note that the first two examples can be combined into $L_{\lambda,\omega}(Q_\kappa)$. More concrete examples are provided by Baldwin, Eklof, and Trlifaj [BET07] and by Zilber's work on pseudoexponentiation [Zil05] [Zil06].

Although AECs are defined semantically, Shelah proved a presentation theorem in [Sh88]. This can be seen as a generalization of Chang's Representation Theorem from [Cha68] that shows that infinitary logic can be seen as omitting types in a larger language. Shelah's Presentation Theorem says the same for AECs.

Definition 2.1.2. *Let T_1 be a first-order theory, Γ a set of finitary, syntactic T_1-types, and $L \subset L(T_1)$ a language. The pseudoelementary class $PC(T_1, \Gamma, L) = \{M \upharpoonright L : M \vDash T_1 \text{ and omits each } p \in \Gamma\}$ for a theory T_1, a set of $L(T_1)$ types Γ, and $L \subseteq L(T_1)$. To say that K is a $PC_{\lambda,\kappa}$ class means that $K = PC(T_1, \Gamma, L)$ for $|T_1| \leq \lambda$ and $|\Gamma| \leq \kappa$.*

Theorem 2.1.3 (Shelah's Presentation Theorem). *Suppose K is an AEC with $LS(K) = \kappa$. There is some $L_1 \supseteq L(K)$ of size κ, a first-order theory T_1 in L_1 of size κ, and a set of L_1-types Γ over the empty set (so $|\Gamma| \leq 2^\kappa$) such that $K = PC(T_1, \Gamma, L(K))$ and for any $M_1 \vDash T_1$ and $N_1 \subseteq M_1$, if M_1 omits Γ, then $N_1 \upharpoonright L(K) \prec_K M_1 \upharpoonright L(K)$. Moreover, every $M \in K$ has an expansion to an $L(K)$ structure $M_1 \in EC(T_1, \Gamma)$ such that, for all N that is an $L(K)$ structure,*

$$N \prec_K M \iff \text{there is some } N_1 \subseteq M_1 \text{ such that } N = N_1 \upharpoonright L(K)$$

Unfortunately, while Chang's theorem provides a very natural connection between the original class and the expanded language, Shelah's theorem offers no natural interpretation for the types. Additionally, it is not often a productive means of investigation when analyzing AECs. The biggest exception to this is Ehrenfeucht-Mostowski models, which function similarly to the first-order case and are a direct consequence of Shelah's Presentation Theorem and previous results on PC classes. We also use the Presentation Theorem in this thesis to prove Łoś' Theorem for AECs (Theorem 4.2.3), which is one of the key results of Chapter IV.

The following definitions are useful for discussing AECs.

Definition 2.1.4.   *1. Given $M, N \in K$, we say $f : M \to N$ is a K-embedding iff f is an injective $L(K)$-morphism such that $f(M) \prec N$. Whenever we write a function between two models, this means that it is a K-embedding.*

2. Given $\lambda \geq LS(K)$, $K_{\leq\lambda} := \{M \in K : \|M\| \leq \lambda\}$. $K_{<\lambda}$, $K_{\geq\lambda}$, and other variations are defined similarly.

3. A resolution of $M \in K$ is a $\prec_K$-increasing, continuous chain $\langle M_i \in K : i < \|M\|\rangle$ such that $M = \cup_{i<\|M\|}M_i$ and $\|M_i\| = |i| + LS(K)$.

Note that the Tarski-Vaught Chain Axioms are, as their name implies, only directly relevant for chains of models. However, this can be strengthened. First, using the closure under isomorphisms, it can be shown that AECs are in fact closed under direct limits of coherent systems.

Fact 2.1.5. *If we have $\langle M_i \in K : i < \kappa \rangle$ and, for $i < j < \kappa$, a coherent set of embeddings $f_{i,j} : M_j \to M_i$—that is, one so, for $i < j < k < \kappa$, $f_{i,k} = f_{j,k} \circ f_{i,j}$—then there is an $L(K)$ structure $M = \varinjlim_{i<j<\kappa}(M_i, f_{i,j})$ and embeddings $f_{i,\infty} : M_i \to M$ such that, for all $i < j < \kappa$, $f_{i,\infty} = f_{j,\infty} \circ f_{i,j}$ and, for each $x \in M$, there is some $i < \kappa$ and $m \in M_i$ such that $f_{i,\infty}(m) = x$. Furthermore, the model M is in K and each $f_{i,\infty}$ is a K-embedding.*

A proof of this fact can be found in [Gro1X]. This first appeared for AECs in VanDieren's thesis [Van02] based on work of Cohn in 1965 on the direct limits of algebras. Second, by inducting on the size of the system, it can also be shown that AECs are also closed under unions of directed systems that are not necessarily linearly ordered.

Fact 2.1.6. *If $(I, <)$ is a directed system and $\{M_i \in K : i \in I\}$ is a collection of models such that $i < j$ implies $M_i \prec_K M_j$, then $\cup_{i\in I}M_i \in K$. Furthermore, if we had $M_i \prec_K N$ for all $i \in I$, then $\cup_{i\in I}M_i \prec_K N$.*

This second statement is more useful than it might appear at first glance. It allows an end segment of an AEC to be completely recovered from just a slice of the AEC at a single cardinal. More formally, call a class $(K, \prec_K)$ an *AEC in λ* if it satisfies all of the AEC axioms from Definition 2.1.1, except that the chain axioms are restricted to chains of length $< \lambda^+$ and K only consists of models of size λ. Note that this makes the Löwenheim-Skolem axiom meaningless. This is the slice we wish to use to construct an AEC with Löwenheim-Skolem number λ.

Definition 2.1.7 ([Sh:h].II.§.23)**.** *Let $(K, \prec_K)$ be an AEC in λ. We define $(K^{up}, \prec^{up})$ by*

- $K^{up} = \{M : M$ is an $L(K)$-structure and there is a directed partial order I and a direct system $\langle M_s \in K : s \in I \rangle$ such that $M = \cup_{s \in I} M_s \}$

- $M \prec^{up} N$ iff there are directed partial orders $I \subset J$ and a direct system $\langle M_s \in K : s \in J \rangle$ such that $M = \cup_{s \in I} M_s$ and $N = \cup_{s \in J} M_s$.

This is denoted K^{up} because it can recover all of the AEC that is "up above" the given slice.

Proposition 2.1.8 ([Sh:h].II.§.23). *1. If $(K, \prec_K)$ is an AEC in λ, then $(K^{up}, \prec^{up})$ is an AEC such that $LS(K^{up}) = \lambda$.*

2. *If $(K, \prec_K)$ is an AEC and $\lambda \geq LS(K)$, then $(K_\lambda, \prec_K)$ is an AEC in λ and*

$$(K_{\geq \lambda}, \prec_K) = ((K_\lambda)^{up}, \prec^{up})$$

This is useful because it helps local analyses to have global consequences. This is key to Shelah's project of good λ-frames (Definition 2.3.5 below), which is taken up in Chapter VI.

We now introduce some of the basic definitions for AECs. We follow the literature by dropping the subscript on the strong substructure (writing $\prec$ for $\prec_K$) and by referring to the AEC by just the class of models (writing K for $(K, \prec_K)$). We also write $\|M\|$ to denote the cardinality of the universe of M; the intention of this notation is to reserve $|M|$ for the universe of M, but we do this only when it clarifies confusing notation.

Definition 2.1.9. *Let K be an AEC.*

1. *Given $LS(K) \leq \lambda \leq \mu, \kappa$, we say K has the (λ, μ, κ)-amalgamation property iff, for all $M_0 \in K_\lambda$, $M_1 \in K_\mu$, and $M_2 \in K_\kappa$ such that $M_0 \prec M_1, M_2$, there is some $N \in K$ and $f_\ell : M_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0$. Having the λ-amalgamation property means having the $(\lambda, \lambda, \lambda)$-amalgamation property and having the amalgamation property means having the (λ, μ, κ)-amalgamation property for all $\lambda, \mu, \kappa \geq LS(K)$. We often shorten "amalgamation property" to just AP.*

2. *Given $LS(K) \leq \mu, \kappa$, we say K has the (μ, κ)-joint mapping property iff for all $M_1 \in K_\mu$ and $M_2 \in K_\kappa$, there is some $N \in K$ and $f_\ell : M_\ell \to N$ for $\ell = 1, 2$. Having the λ-joint mapping property means having the (λ, λ)-joint mapping property and having the joint mapping property means having the (μ, κ)-joint mapping property for all $\mu, \kappa \geq LS(K)$. We often shorten "joint mapping property" to just JMP. This is sometimes called the joint embedding property.*

3. *K has no maximal models iff, for every $M \in K$, there is $N \in K$ such that $M \precneqq N$.*

4. *K has arbitrarily large models iff, for every $\lambda \geq LS(K)$, there is $N \in K_\lambda$.*

The properties listed above all hold of complete elementary classes: no maximal models and arbitrarily large models follow easily from the compactness theorem and amalgamation and joint mapping follow from compactness and interpolation/Robinson's Consistency Lemma. Note that the amalgamation property is defined assuming the base is a model. In complete first-order theories, a stronger property of amalgamation over sets holds. However, this seems rare outside of elementary classes: even Shelah [Sh87a] [Sh87b] only proves amalgamation over good sets. Another stronger form of amalgamation holds in elementary classes: disjoint amalgamation.

Definition 2.1.10. *An AEC K has* disjoint amalgamation *iff, for all $M_0, M_1, M_2 \in K$ such that $M_0 \prec M_1, M_2$, there is some $N \in K$ and $f_\ell : M_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0$ and $f(M_1) \cap f_2(M_2) = f_1(M_0)$.*

There are several well known relationships between these properties.

Proposition 2.1.11. *Let K be an AEC and $LS(K) \leq \lambda \leq \mu, \kappa$.*

1. *(λ, μ, κ)-AP implies (λ, μ, κ^+)-AP.*

2. *If λ-AP holds for all $\lambda \geq LS(K)$, then AP holds.*

3. *AP and λ-JMP imply that $K_{\geq \lambda}$ has JMP.*

4. *No maximal models implies arbitrarily large models.*

5. *Arbitrarily large models and JMP imply no maximal models.*

6. *Categoricity in λ implies λ-JMP.*

7. *λ-JMP and no maximal models imply $K_{\leq \lambda}$ has JMP.*

An assumption of amalgamation, joint mapping, and no maximal models[1] has become common in many works on AECs. Historically, this can be traced back to Shelah's work in [Sh394], where a global amalgamation assumption and categoricity at a successor above the second Hanf number were used to great effect; Baldwin [Bal09] provides an excellent account of this argument. A more mathematical impetus is that these three assumption together allow the construction of a universal domain or monster model, here denoted by $\mathfrak{C}$. Formally, this is a special model (see [Gro1X].4.4) of cofinality well above any cardinalities under consideration. Informally, we use it as a highly universal and homogeneous model that is assumed to contain all models under consideration. The usefulness of the monster model truly shines when discussing Galois types, which we do now.

There are two ways of defining Galois types, one for the general context (although these are greatly simplified under amalgamation) and one in the context of a monster model. Obviously, these two definitions are equivalent with a monster model.

Definition 2.1.12. *Let K be an AEC, $\lambda \geq LS(K)$, and I be a set.*

1. *Set $K_\lambda^{3,I} = \{(\langle a_i : i \in I \rangle, M, N) : M \in K_\lambda, M \prec N \in K_{\lambda+|I|}, \text{ and } \{a_i : i \in I\} \subset |N|\}$. The elements of this set are referred to as* pretypes.

2. *Given two pretypes $(\langle a_i : i \in I \rangle, M, N)$ and $(\langle b_i : i \in I \rangle, M', N')$ from $K_\lambda^{3,I}$, we say that $(\langle a_i : i \in I \rangle, M, N) \sim_{AT} (\langle b_i : i \in I \rangle, M', N')$ iff $M = M'$ and there is $N^* \in K$ and $f : N \to N^*$ and $g : N' \to N^*$ such that $f(a_i) = g(b_i)$ for all $i \in I$ and the following diagram commutes:*

$$
\begin{array}{ccc}
N' & \xrightarrow{g} & N^* \\
\uparrow & & \uparrow f \\
M & \longrightarrow & N
\end{array}
$$

Note that 'AT' denotes 'atomic.'

---

[1] Equivalently, this can be replaced by arbitrarily large models

3. *Let $\sim$ be the transitive closure of $\sim_{AT}$.*

4. *If $\mathfrak{C}$ is a monster model, $M \prec \mathfrak{C}$, and $\langle a_i : i \in I \rangle$ and $\langle b_i : i \in I \rangle$ are elements from $\mathfrak{C}$, then*

$$gtp(\langle a_i : i \in I \rangle / M) = gtp(\langle b_i : i \in I \rangle / M)$$

   iff there is $f \in Aut_M \mathfrak{C}$ such that $f(a_i) = b_i$ for all $i \in I$.

5. *For $M \in K$, set $gtp(\langle a_i : i \in I \rangle / M, N) = [(\langle a_i : i \in I \rangle, M, N)]_\sim$ and $gS^I(M) = \{gtp(\langle a_i : i \in I \rangle / M, N) : (\langle a_i : i \in I \rangle / M, N) \in K_{\|M\|}^{3,I}\}$.*

6. *For $M \in K$, define $gS_{na}^I(M) = \{gtp(\langle a_i : i \in I \rangle / M, N) \in S^I(M) : a_i \in N - M \text{ for all } i \in I\}$.*

It should be noted that amalgamation implies that λ-AP implies that $\sim_{AT}$ is already transitive and, thus an equivalence relation.[2] Also, the definition of Galois types in the monster model hints to the choice of the naming of Galois type from Grossberg [Gro02]. We continue to call these 'Galois types' in this chapter and to use the 'g' in 'gtp' above and 'gS' below. However, in subsequent chapters, we follow the literature by simply referring to them as 'types' and removing the 'g' when there is no confusion with other notions of type.

Note that we have only defined Galois types over models; this is by design. Some early definitions of Galois types (as in [Sh394]) and work of Hyttinen and Kesälä allow for Galois types over sets as amalgamation over models is enough to make $\sim_{AT}$ transitive here. However, more advanced constructions like Galois-saturated or universal models (defined below) would require amalgamation over sets to work for this expanded definition of type. Even something as simple as extending a type requires amalgamation over the base.

Definition 2.1.13. *$M \in K$ is an amalgamation base iff, for every $N_1, N_2 \in K$ such that $M \prec N_1$ and $M \prec N_2$, there is some $N \in K$ and $f_\ell : M_\ell \to N$ for $\ell = 1, 2$ such that $f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0$.*

Proposition 2.1.14 (Boney-Vasey)**.** *Suppose K has λ-AP. Let A be a set. Then A is an amalgamation base iff, for every $p \in gS(A)$ and $A \subset M$, p has an extension to M.*

Galois types are very useful in analyzing AECs, as evidenced by their wide use in the literature at large and in this thesis. The author's informal take on this is that Galois types allow model theorists to import more of their model-theoretic intuition from the first-order context to AECs; this notion will return in the next section. While more nuanced concepts can't be transferred directly, many ideas defined in terms of types can be defined in AECs in essentially the same way as first order. Thus, we offer the following definitions of first-order concepts, tweaked for the present context.

Definition 2.1.15. *Let K be an AEC and $M \in K$.*

1. *We say that M is κ-Galois saturated iff, for every $N \prec M$ such that $\|N\| < \kappa$ and every $p \in gS^1(N)$, p is realized in M. We say that M is Galois saturated iff it is $\|M\|$-Galois saturated.*

2. *We say that M is κ-model homogeneous iff, for every $N \prec M$ and $N' \succ N$ such that $\|N'\| < \kappa$, there is some $f : N' \to_N M$. We say that M is model homogeneous iff it is $\|M\|$-model homogeneous.*

---

[2]More precisely, $\lambda + |I|$-AP implies that $\sim_{AT}$ is transitive on $K_\lambda^{3,I}$.

3. *We say that M is κ-universal over N iff, for all $N' \succ N$ such that $\|N'\| < \kappa$, there is $f : N' \rightarrow_N M$, allowing N to be empty. We say that M is* universal over N *iff it is $\|M\|$-universal over N.*

We now state the "model-homogeneity = saturation" lemma for AECs. This has long been known for first-order theories and first appeared for AECs in [Sh300], although a correct proof was not given in print until Shelah [Sh576].0.26.1.

Lemma 2.1.16 (Shelah)**.** *Let K be an AEC with amalgamation and $\lambda > LS(K)$. Then the following are equivalent for $M \in K$:*

- *M is λ-model homogeneous: for every $N_1 \prec N_2 \in K_{<\lambda}$ with $N_1 \prec M$, there is a K embedding $f : N_2 \rightarrow_{N_1} M$; and*

- *M is λ-Galois saturated: for every $N \prec M$ with $\|N\| < \lambda$ and every $p \in S^1(N)$, p is realized in M.*

Limit models (introduced as (λ, α)-saturated in [KoSh362], see the definition below) have been suggested as a substitute for saturated models and the question of uniqueness of limit models has been suggested as a dividing line for AECs; see Shelah [Sh576].

Definition 2.1.17.   *1. Let $M \in K_\lambda$ and $\alpha < \lambda^+$ be a limit ordinal. N is (λ, α)-limit over M iff there N has a resolution $\langle N_i \in K_\lambda : i < \alpha \rangle$ such that $N_0 = M$ and M_{i+1} is universal over M_i.*

2. *K has unique limit models in λ when, if $M, N_1, N_2 \in K_\lambda$ and $\alpha_1, \alpha_2 < \lambda^+$ such that N_ℓ is (λ, α_ℓ)-limit over M, then $N_1 \cong_M N_2$.*

It is an easy exercise to show that (2) holds if cf $\alpha_1 = $ cf α_2.

2.2 Tameness and Type Shortness

Suppose that p and q are Galois types and we wish to determine if $p = q$. This is, of course, a very general question, but there are two features of p and q that must align if it is possible for them to be equal: they must be over the same model and realizations of them must have the same index structure. We call these properties the domain and length of the type.

Definition 2.2.1. *Let $p \in gS^I(M)$. Then*

- *the* domain of p is M, *denoted dom p; and*

- *the* length of p is I, *denoted $\ell(I)$.*

We call I the length of p even though it need not have any particular structure. However, we could always "rearrange" the type and assume its length was a cardinal or ordinal.

Given these two parameters, we can form the restrictions of types.

Definition 2.2.2. *Let $p = gtp(\langle a_i : i \in I \rangle / M; N) \in gS^I(M)$.*

- *If $M_0 \prec M$, then $p \upharpoonright M_0 = gtp(\langle a_i : i \in I \rangle / M_0; N) \in gS^I(M_0)$.*

- *If $I_0 \subset I$, then $p^{I_0} = gtp(\langle a_i : i \in I_0 \rangle / M; N) \in gS^{I_0}(M)$.*

It is easy to see that this does not depend on the choice of pretype representing p and that $p^{I_0} \upharpoonright M_0 = (p \upharpoonright M_0)^{I_0}$, although we will always write the former.

We now return to the question of when $p = q$. Again, this question is too general to answer. However, in the first-order context, there is a local test to determine this. That is, if $p \neq q$, then this is witnessed by some $\phi(\mathbf{x}, \mathbf{b}) \in p - q$. This means that there is a restriction of p and q to a finite domain and length that already sees this difference. Contrapositively, in the first-order context, if two types are equal on all finite restrictions, then the types themselves are equal. One can ask if there is a similar decision procedure for AECs, e.g. does it suffice to check equality on all small restrictions?

If the reader has been through the introduction, then the answer is already known to be no. However, the reader then also knows the importance of tameness and type shortness, the eponymous properties of this section that imply that there are such procedures for the domain and the length respectively. Tameness is a property first isolated by Rami Grossberg and Monica VanDieren in [GV06b]. The property is similar to one used by Shelah in [Sh394], where he derived this property for types with *saturated* domains from categoricity in a successor cardinal above the second Hanf number, $\beth_{(2^{\beth_{(2^{LS(K)})^+}})^+}$; Shelah's property is now called *weak tameness* (see [BKV06]). In their papers, Grossberg and VanDieren defined only χ-tameness; the two cardinal parameterization of it appeared later in [Bal09].

We begin with a minor notational definition and then define several levels of tameness:

Definition 2.2.3. *Suppose K is an AEC with $LS(K) < \kappa \leq \lambda$ and I is a linear order.*

1. *For any $M \in K_{\geq \kappa}$, we write*[3]

$$P_\kappa^* M = \{N \prec M : \|N\| < \kappa\}$$

2. *K is $(< \kappa, \lambda)$-tame for I-length types iff for any $M \in K_\lambda$ and $p \neq q \in S^I(M)$, there is some $N \in P_\kappa^* M$ and $p \upharpoonright N \neq q \upharpoonright N$.*

3. *K is $< \kappa$-tame for I-length types iff K is $(< \kappa, \mu)$ tame for I-length types for all $\mu \geq \kappa$.*

4. *K is fully $< \kappa$-tame iff K is $< \kappa$-tame for I-length types for all I.*

5. *Writing "κ" for "$< \kappa$" means "$< \kappa^+$."*

If we omit the I, we mean $I = 1$.

Type shortness is the natural dual to tameness and was isolated by the author in [Bona], the bulk of which appears as Chapter IV in this thesis.

Definition 2.2.4. *Suppose K is an AEC with $LS(K) \leq \mu$ and $\kappa < \lambda$.*

1. *K is $(< \kappa, \lambda)$-type short over μ-sized models iff for any $M \in K_\mu$ and $p \neq q \in S^\lambda(M)$, there is some $I' \subset I$ of size $< \kappa$ such that $p^{I'} \neq q^{I'}$.*

2. *K is $< \kappa$-type short over μ-sized models iff K is $(< \kappa, \lambda)$-type short over μ-sized models for all $\lambda \geq \kappa$.*

3. *K is fully $< \kappa$-type short iff K is $< \kappa$-type short over μ-sized models for all μ.*

---

[3]$P_\kappa^* M$ is reminiscent of the set theoretic notation $P_\kappa A = \{X \subset A : |X| < \kappa\}$.

4. Writing "κ" for "$< \kappa$" means "$< \kappa^+$."

While this has been isolated more recently than tameness, type shortness has already proved to be of worth by being a key feature in the development of the nonforking relation in Chapter V.

We describe these properties as dual based on their definition. However, by interchanging domains and lengths, we get the following implications between them. In particular, type shortness appears to be a stronger property than tameness.

Theorem 2.2.5. *Suppose K is an AEC with amalgamation, joint embedding, and no maximal models. If K is categorical in μ and $(< \kappa, \mu)$-tame for λ-length types, then K is $(< \kappa, \mu)$-type short for types of models over λ-sized domains.*

Proof: Let $M, M' \in K_\mu$ and $N \in K_\lambda$ such that $tp(M/N) \neq tp(M'/N)$. By μ categoricity, there is some $f : M \cong M'$; WLOG $f \in Aut\mathfrak{C}$.

Claim: $tp(f(N)/M') \neq tp(N/M')$

If not, then there is some $h \in Aut_{M'}\mathfrak{C}$ such $h \circ f(N) = N$. Then $h \circ f \in Aut_N\mathfrak{C}$ and $h \circ f(M) = h(M') = M'$, which means $tp(M/N) = tp(M'/N)$, a contradiction. $\dagger_{\text{Claim}}$

Now, by tameness, there is some $M^- \in P_\kappa^* M'$ such $tp(f(N)/M^-) \neq tp(N/M^-)$. Then, by the same argument as in the claim, we get that $tp(f^{-1}(M^-)/N) \neq tp(M^-/N)$, which is what we want because $f^{-1}(M^-) \in P_\kappa^* M$. $\dagger$

Theorem 2.2.6. *Suppose K is an AEC with amalgamation, joint embedding, and no maximal models. If K is $(< \kappa, \mu)$-type short over the empty set, then it is $(< \kappa, \mu)$-tame for $\leq \mu$-length types.*

Proof: Suppose $tp(a/M) \neq tp(b/M)$ for $M \in K_\mu$ and $\ell(a) = \ell(b) \leq \mu$. Then we have $tp(aM/\emptyset) \neq tp(bM/\emptyset)$. By our type shortness, there is some $a' \subset a$, $b' \subset b$, and $X_0 \subset M$ all of cardinality $< \kappa$ such that $tp(a'X_0/\emptyset) \neq tp(b'X_0/\emptyset)$. Then find $M_0 \prec M$ of size $< \kappa$ that contains X. Then

$$
\begin{aligned}
tp(a'X_0/\emptyset) &\neq tp(b'X_0/\emptyset) \\
tp(aM_0/\emptyset) &\neq tp(bM_0/\emptyset) \\
tp(a/M_0) &\neq tp(b/M_0)
\end{aligned}
$$

as desired. $\dagger$

While there are many ways of combining these properties, we focus on AECs that are fully $< \kappa$-tame and fully $< \kappa$-type short; this is the conclusion we get from the strongest result of Chapter IV (Theorem 4.2.5) and is the hypothesis for the main theorem of Chapter V (Theorem 5.3.1). In this context, any two types that differ already differ on some domain and length of size $< \kappa$; in this case, we call domains and lengths of size $< \kappa$ "small." With a little imagination, this is similar to the first-order setting when "finite" is replaced by "small." Although this seems like a complete change of context, it gives rise to an intuition where small types can be treated as formulas. While this analogy has its flaws–it is hard to build types inductively, restricting to sets of formulas is impossible, etc–this intuition is strong enough to adapt many first-order arguments and definitions to the $< \kappa$-tame and -type short context.

2.3 Nonforking in Abstract Elementary Classes

Since the publication of the first edition of Shelah's book [Sh:a], nonforking has become widely used in model theory and is the key tool in classification thoery. We discuss this and different attempts to find the right notion of nonforking in AECs in the introduction; here we provide the relevant definitions.

The so-called anchor symbol $\downarrow$ is the one commonly used for nonforking and we continue that here. More specifically, $\downarrow$ is used for concrete nonforking relations (coheir in Chapter V and a good λ-frame in Chapter VI), while we use $\overset{*}{\downarrow}$ anytime we wish to discuss an abstract independence relation. We write $A \overset{*}{\underset{M}{\downarrow}} \widehat{M} N$ or $\overset{*}{\downarrow}(M, N, A; \widehat{M})$ to mean "$gtp(A/N; \widehat{M})$ does not fork over M."[4] Again, in a departure from first-order, the base and right-hand inputs are required to be models; this is connected to our requirement that types have models as domains. Work with Grossberg, Kolesnikov, and Vasey [BGKV] develops the notion of a closure nonforking relation that weakens the requirement on the right hand model; however, we do not explore this in this thesis. Also, when working with nonforking relations in the context of a monster model (as in Chapter V), the specification of the ambient model $\widehat{M}$ is unnecessary as we can always take $\widehat{M} = \mathfrak{C}$. Thus, we omit it.

The following list of properties is the ideal list of properties that an independence relation might have. It corresponds to the properties of nonforking in stable first-order theories, although the naming and presentation draw on Makkai and Shelah's work. The main departure from this is Symmetry, which, stemming from the necessity that certain inputs be models, comes from Shelah's definition in good λ-frames; see Definition 2.3.5 below. Also, in order to present a finer analysis in Chapter IV, we parameterize the properties of Existence, Extension, Uniqueness, and Symmetry. The order of these parameters is designed to be as uniformized as possible: the λ refers to the size of the left object, μ refers to the size of the middle object, and χ refers to the size of the right object. If we write a property without parameters, then we mean that property for all possible parameters.

Definition 2.3.1. *Fix an AEC K. Let $\overset{*}{\downarrow}$ be a ternary relation on models and sets such that $A \overset{*}{\underset{M_0}{\downarrow}} N$ implies that A is a subset of the monster model and $M_0 \prec N$ are both models. We say that $\overset{*}{\downarrow}$ is an* independence relation *iff it satisfies all of the following properties for all cardinals referring to sets and all cardinals that are at least κ when the cardinal refers to a model.*

(I) **Invariance**

Let $f \in Aut\, \mathfrak{C}$ be an isomorphism. Then $A \overset{*}{\underset{M_0}{\downarrow}} N$ implies $f(A) \overset{*}{\underset{f(M_0)}{\downarrow}} f(N)$.

(M) **Monotonicity**

If $A \overset{*}{\underset{M_0}{\downarrow}} N$ and $A' \subset A$ and $M_0 \prec M_0' \prec N' \prec N$,, then $A' \overset{*}{\underset{M_0'}{\downarrow}} N'$.

(T) **Transitivity**

If $A \overset{*}{\underset{M_0'}{\downarrow}} N$ and $M_0' \overset{*}{\underset{M_0}{\downarrow}} N$ with $M_0 \prec M_0'$, then $A \overset{*}{\underset{M_0}{\downarrow}} N$.

(C)$_{<\kappa}$ **Continuity**

---

[4]Technically, we need $\overset{*}{\downarrow}$ to satisfy Invariance and Monotonicity (defined below) for this to be well-formed. However, all nonforking relations under consideration have these properties so there is no confusion.

a) *If for all small $A' \subset A$ and small $N' \prec N$, there is $M_0' \prec M_0$ and $N' \prec N^* \prec N$ such that $M_0' \prec N^*$ and $A' \underset{M_0'}{\overset{*}{\downarrow}} N^*$, then $A \underset{M_0}{\overset{*}{\downarrow}} N$.*

b) *If $\langle A_i, M_0^i \mid i < \kappa \rangle$ are filtrations of A and M_0 and $A_i \underset{M_0^i}{\overset{*}{\downarrow}} N$ for all $i < \kappa$, then $A \underset{M_0}{\overset{*}{\downarrow}} N$.*

$(E)_{(\lambda,\mu,\chi)}$ a) **Existence**

Let A be a set and M_0 be a model sizes λ and μ, respectively. Then $A \underset{M_0}{\overset{*}{\downarrow}} M_0$.

b) **Extension**

Let A be a set and M_0 and N be models of sizes λ, μ, and χ, respectively, such that $M_0 \prec N$ and $A \underset{M_0}{\overset{*}{\downarrow}} N$. If $N^+ \succ N$ of size χ, then there is A' such that $A' \underset{M_0}{\overset{*}{\downarrow}} N^+$ and $tp(A'/N) = tp(A/N)$.

$(S)_{(\lambda,\mu,\chi)}$ **Symmetry**

Let A_1 be a set, M_0 be a model, and A_2 be a set of sizes λ, μ, and χ, respectively, such that there is a model M_2 with $M_0 \prec M_2$ and $A_2 \subset |M_2|$ such that $A_1 \underset{M_0}{\overset{*}{\downarrow}} M_2$. Then there is a model $M_1 \succ M_0$ that contains A_1 such that $A_2 \underset{M_0}{\overset{*}{\downarrow}} M_1$.

$(U)_{(\lambda,\mu,\chi)}$ **Uniqueness**

Let A and A' be sets and $M_0 \prec N$ be models of sizes λ, λ, μ, and χ, respectively. If $tp(A/M_0) = tp(A'/M_0)$ and $A \underset{M_0}{\overset{*}{\downarrow}} N$ and $A' \underset{M_0}{\overset{*}{\downarrow}} N$, then $tp(A/N) = tp(A'/N)$.

The axioms $(E)_{(\lambda,\mu,\chi)}$ combines two notions. The first is Existence: that a type does not fork over its domain. This is similar to the consequence of simplicity in first order theories that a type does not fork over the algebraic closure of its domain. As mentioned above, in this context, existence is equivalent to every model being κ saturated. In the first order case, where finite satisfiability is the proper analogue of our nonforking, existence is an easy consequence of the elementary substructure relation. In [MaSh285], this holds for $< \kappa$ satisfiability, their nonforking, because types are formulas from $L_{\kappa,\kappa}$ and, due to categoricity, the strong substructure relation is equivalent to $\prec_{L_{\kappa,\kappa}}$.

The second notion is the extension of nonforking types. In first-order theories (and in [MaSh285]), this follows from compactness but is more difficult in a general AEC. We have separated these notions for clarity and consistency with other sources, but could combine them in the following statement:

Let A be a set and M_0 and N be models of sizes λ, μ, and χ, respectively, such that $M_0 \prec N$. Then there is some A' such that $tp(A'/M_0) = tp(A/M_0)$ and $A' \underset{M_0}{\overset{*}{\downarrow}} N$.

As an alternative to assuming (E), and thus assuming all models are κ saturated, we could simply work with the definition and manipulate the nonforking relationships that occur. This is the strategy in Section 5.4. In such a situation, κ saturated models, which will exist in $\lambda^{<\kappa}$, will satisfy the existence axiom.

The relative complexity of the symmetry property is necessitated by the fact that the right side object is required to be a model that contains the base. If the left side object already satisfied this, then there is a simpler statement.

Proposition 2.3.2. *If $(S)_{(\lambda,\mu,\chi)}$ holds, then so does the following*

$(S^*)_{(\lambda,\mu,\chi)}$ *Let M, M_0, and N be models of size λ, μ, and χ, respectively such that $M_0 \prec N$ and $M_0 \prec M$.*
Then $M \underset{M_0}{\overset{*}{\downarrow}} N$ iff $N \underset{M_0}{\overset{*}{\downarrow}} M$.

In first order stability theory, many of the key dividing lines depend on the local character $\kappa(T)$, which is the smallest cardinal such that any type doesn't fork over some subset of its of domain of size less than $\kappa(T)$. The value of this cardinal can be smaller than the size of the theory, e.g. in an uncountable, superstable theory. However, since types and nonforking occur only over models, the smallest value the corresponding cardinal could take would be $LS(K)^+$. This is too coarse for many situations. Instead, we follow [ShVi635], [Sh:h].II, [GV06b], and [GVV] by defining a local character cardinal based on the length of a resolution of the base rather than the size of cardinals. As different requirements appear in different places, we give two definition of local character: one with no additional requirement, as in [Sh:h].II, and one requiring that successor models be universal, as in [ShVi635], [GV06b], and [GVV].

Definition 2.3.3. $\kappa_\alpha(\overset{*}{\downarrow}) = \min\{\lambda \in REG \cup \{\infty\} : \text{for all } \mu = cf\,\mu \geq \lambda \text{ and all increasing, continuous}$
chains $\langle M_i : i < \mu \rangle$ and all sets A of size less than α, there is some $i_0 < \mu$ such that $A \underset{M_{i_0}}{\overset{*}{\downarrow}} \cup_{i<\mu} M_i\}$

$\kappa_\alpha^*(\overset{*}{\downarrow}) = \min\{\lambda \in REG \cup \{\infty\} : \text{for all } \mu = cf\,\mu \geq \lambda \text{ and all increasing, continuous chains}\langle M_i :$
$i < \mu \rangle$ *with M_{i+1} universal over M_i and κ saturated and all sets A of size less than α, there is some*
$i_0 < \mu$ *such that $A \underset{M_{i_0}}{\overset{*}{\downarrow}} \cup_{i<\mu} M_i\}$*
In either case, if we omit α, then we mean $\alpha = \omega$.

In Section 5.4, we return to these properties and examine natural conditions that imply that $\kappa^*(\downarrow) = \omega$. The main use of these concepts in this thesis is in Chapter V, where we, in joint work with Grossberg, introduce and develop a nonforking relation for short and tame AECs that generalizes coheir from stable first-order theories and $< \kappa$-satisfiability from Makkai and Shelah [MaSh285]. We use this definition in the context of a monster model, so we omit the ambient model.

Definition 2.3.4. *Let $M_0 \prec N$ be models and A be a set. We say that $tp(A/N)$ does not fork over M_0, written $A \underset{M_0}{\downarrow} N$, iff for all small $a \in A$ and all small $N^- \prec N$, we have that $tp(a/N^-)$ is realized in M_0.*

We also give the definition for good λ-frames. With some effort, the definitions above could be generalized to include the definition of frame as a special case. However, we avoid this to allows easier comparison with the existing literature. Informally, a good λ-frame $\mathfrak{s}$ consists of an AEC in λ denoted K_λ; a nonforking relation $\underset{\mathfrak{s}}{\downarrow}$ on K_λ; and, for each model $M \in K_\lambda$, a collection of nonalgebraic 1-types $S_{\mathfrak{s}}^{bs}(M)$ called *basic types* on which $\underset{\mathfrak{s}}{\downarrow}$ operates. Stronger than the properties given above, good λ-frames generalize nonforking in superstable theories along with the collection of regular types.

Definition 2.3.5. $\mathfrak{s} = (K_\lambda, \underset{\lambda}{\downarrow}, S_\lambda^{bs}) = (K^{\mathfrak{s}}, \underset{\mathfrak{s}}{\downarrow}, S_{\mathfrak{s}}^{bs})$ *is a good λ-frame iff*

(A) K_λ *is an AEC in λ (we denote this cardinal with $\lambda_{\mathfrak{s}}$);*

(C) K_λ *has AP, JMP, and no maximal models;*

(D) (a) $S_{\mathfrak{s}}^{bs}(M) \subset S(M)$, *the domain of $S_{\mathfrak{s}}^{bs}$ is K_λ, and it respects isomorphisms;*

(b) $S_{\mathfrak{s}}^{bs}(M) \subset S^{na}(M)$;

(c) **Density:** if $M \not\succeq N$ from K_λ, then there is some $a \in N - M$ such that $tp(a/M, N) \in S_{\mathfrak{s}}^{bs}(M)$;

(d) bs-**stability:** $|S_{\mathfrak{s}}^{bs}(M)| \leq \lambda$ for all $M \in K_\lambda$;

(E) (a) **Invariance:** $\underset{\lambda}{\bigcup} = \underset{\mathfrak{s}}{\bigcup} = \bigcup$ is a four-place relation in which the first, second, and fourth inputs are models from K_λ and the third input is an element such that $\bigcup(M_0, M_1, a, M_3)$[5] is preserved under isomorphisms and implies i) $M_0 \prec M_1 \prec M_3$; ii) $a \in M_3 - M_1$; and iii) $\bigcup(M_0, M_0, a, M_3)$ is equivalent to $tp(a/M_0, M_3) \in S_{\mathfrak{s}}^{bs}(M_0)$;

(b) **Monotonicity:** if $M_0 \prec M_0' \prec M_1' \prec M_1 \prec M_3'' \prec M_3 \prec M_3'$ and $a \in M_3''$, then $\bigcup(M_0, M_1, a, M_3)$ implies $\bigcup(M_0, M_1, a, M_3'')$ and $\bigcup(M_0', M_1', a, M_3')$;

(c) **Local Character:** if $\langle M_i \in K_\lambda : i \leq \delta + 1 \rangle$ is increasing, continuous, $a \in M_{\delta+1}$, and $tp(a/M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta)$, then there is some $i_0 < \delta$ such that $\bigcup(M_i, M_\delta, a, M_{\delta+1})$;

(e) **Uniqueness:** If $p, q \in S^{bs}(M_1)$ do not fork over $M_0 \prec M_1$ and $p \upharpoonright M_0 = q \upharpoonright M_1$, then $p = q$;

(f) **Symmetry:** If $M_0 \prec M_1 \prec M_3$, $a_1 \in M_1$, $tp(a_1/M_0, M_3) \in S^{bs}(M_0)$, and $\bigcup(M_0, M_1, a_2, M_3)$, then there are M_2 and M_3' such that $a_2 \in M_2$, $M_0 \prec M_2 \prec M_3'$, $M_3 \prec M_3'$, and $\bigcup(M_0, M_2, a_1, M_3')$;

(g) **Extension Existence:** If $M \prec N$ and $p \in S^{bs}(M)$, then there is some $q \in S^{bs}(N)$ such that $p \leq q$ and q does not fork over M;

(h) **Continuity:** if $\langle M_i \in K_\lambda : i \leq \delta \rangle$ with δ limit, $p \in M_\delta$, and, for all $i < \delta$, $p \upharpoonright M_i$ does not fork over M_0, then $p \in S^{bs}(M_\delta)$ and p does not fork over M_0.

The strange numbering is inherited from [Sh:h].II, where frames are introduced. The missing axiom (B) is the existence of a superlimit model and is omitted as in [JrSh875]. Axioms (E)(d) and (i) are shown to follow from the other axioms.

Theorem 2.3.6 ([Sh:h].II.§2.18, .16)**.** • *Axioms (A), (C), (D)(a) and (b), and (E)(a), (b), (e), and (g) imply Axiom (E)*

(d) **Transitivity:** if $M_0 \prec M_0' \prec M_0'' \prec M_3$ from K_α and $a \in M_3$, then $\bigcup(M_0, M_0', a, M_3)$ and $\bigcup(M_0', M_0'', a, M_3)$ implies $\bigcup(M_0, M_0'', a, M_3)$.

• *Axioms (A), (C), and (E)(b), (d), (f), and (g) imply Axiom (E)*

(i) **Non-forking Amalgamation:** if, for $\ell = 1, 2$, $M_0 \prec M_\ell$ from K_λ, $a_\ell \in M_\ell - M_0$, and $tp(a_\ell/M_0, M_\ell) \in S^{bs}(M_0)$, then there are f_1, f_2, M_3 such that $M_0 \prec M_3 \in K_\lambda$ and, for $\ell = 1, 2$, we have $f_\ell : M_\ell \to_{M_0} M_3$ and $\bigcup(M_0, f_{3-\ell}(M_{3-\ell}), f_\ell(a_\ell), M_3)$.

The most general theorem on the existence of good λ-frames is the following, which builds on the work of [Sh576].

Theorem 2.3.7 ([Sh:h].II.§3.7)**.** *Assume* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and*

---

[5]We use this notation to follow Shelah. In the previous notation, this would be $a \underset{M_0}{\overset{M_3}{\bigcup}} M_1$.

1. K is an AEC with $LS(K) \leq \lambda$;

2. K is categorical in λ and λ^+;

3. K has a model in λ^{++}; and

4. $I(\lambda^{++}, K) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$ and $WDmId(\lambda^+)$ is not λ^{++}-saturated.

Then there is a good λ^+-frame.

2.4 Ultraproducts and Continuous Logic

The ultraproduct construction is used heavily in this thesis, so we review it here. Note that we do not refer to constants directly; instead, we view them as 0-ary functions.

Definition 2.4.1. *Suppose that I is an index set and U is an ultrafilter on I. Let $\{M_i : i \in I\}$ be a collection of L-structures.*

- *$\Pi M_i := \{f : I \times \cup_{i \in I} M_i : \forall i \in I, f(i) \in M_i\}$*

- *For $f, g \in \Pi M_i$,*
$$fUg \iff \{i \in I : f(i) = g(i)\} \in U$$

- *For $F \in L$ and $[f_0]_U, \ldots, [f_{n-1}]_U \in \Pi M_i/U$,*
$$F^{\Pi M_i/U}([f_0]_U, \ldots, [f_{n-1}]_U) = [i \mapsto F^{M_i}(f_0(i), \ldots, f_{n-1}(i))]_U$$

- *For $R \in L$ and $[f_0]_U, \ldots, [f_{n-1}]_U \in \Pi M_i/U$,*
$$R^{\Pi M_i/U}([f_0]_U, \ldots, [f_{n-1}]_U) \iff \{i \in I : R^{M_i}(f_0(i), \ldots, f_{n-1}(i))\} \in U$$

- *The L-structure $\Pi M_i/U$ is given by the universe, functions, and relations given above.*

Variants of this construction are discussed in Chapters VII, VIII, and IX. When necessary, we distinguish the above construction from those by referring to it as the classic or model-theoretic ultraproduct.

The foundation of any model-theoretic analysis of the ultraproduct is Łoś' Theorem, which connects the first-order behavior of the ultraproduct to the first-order behavior of a U-large set of models.

Theorem 2.4.2 (Łoś). *Suppose that I is an index set, U is an ultrafilter on I, and $\{M_i : i \in I\}$ is a collection of L-structures. If $\phi(x_0, \ldots, x_{n-1}) \in L$ and $[f_0]_U, \ldots, [f_{n-1}]_U \in \Pi M_i/U$, then*

$$\Pi M_i/U \vDash \phi([f_0]_U, \ldots, [f_{n-1}]_U) \iff \{i \in I : M_i \vDash \phi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$

Importantly, this connection is only first-order. In general, ultraproducts have some degree of saturation. Indeed, the study of this saturation, using regular ultrafilters, has lead to the study of Keisler's Order.

Since AECs are often classes of models omitting certain types, this saturation works contrary to our goals. One method of avoiding this saturation is through the use of more complete ultrafilters; this is explored in Chapter IV and the end of Chapter V. Another way to avoid this saturation while working

inside ZFC is to vary the construction itself. As mentioned above, this is done in Chapters VII, VIII, and IX.

The variations introduced are connected to continuous first-order logic. While not an AEC (although Chapter VII shows that it can be represented as one), this is a context beyond classic first-order model theory that has seen much activity in recent years. Continuous logic is based on the idea of a continuous language. Rather than discrete relations and functions, the symbols of continuous logic are uniformly continuous functions that take value in the model (for function symbols) or in $[0, 1]$ (for relation symbols); $[0, 1]$ is the set of truth values in this logic, with 0 representing true and 1 representing false. To make sense of this, the logic replaces equality with a metric and adds moduli of uniform continuity for each function and relation symbol.

As a brief aside, given $f : (M, d) \to (M', d')$, there are two equivalent ways to say that f is uniformly continuous.

Definition 2.4.3.   • $\Delta_f(\epsilon) := \inf\{d(x, y) : x, y \in M, d'(f(x), f(y)) > \epsilon\}$

• $w^f(\delta) := \sup\{d'(f(x), f(y)) : x, y \in M, d(x, y) < \delta\}$

Fact 2.4.4. *f is uniformly continuous iff $\Delta_f(\epsilon) > 0$ for all $\epsilon > 0$ iff $\lim_{\delta \to 0^+} w^f(\delta) = 0$.*

Formulas are built up using any uniformly continuous function from $[0, 1]^n \to [0, 1]$ as connectives and, thus, formulas are also uniformly continuous. However, as discussed in [BBHU08].§6, it suffices to consider a *full* set of connectives and we will assume that the set used is $1 - x$; $\frac{x}{2}$; $x \dot{-} y$; $\inf$; and $\sup$. Structures in continuous logic have universes that are complete with respect to the metric and functions and relations are uniformly continuous with respect to the models.

In order to prove compactness, an analytic variant of the ultraproduct is used. A description can be found in Iovino [Iov02], but we review the construction here. The notation is different to make comparisons with later chapters easier. Let $\{\mathcal{B}_i : i \in I\}$ be a collection of Banach spaces, possibly in an expanded language, and U be an ultrafilter. Let $\Pi^*\mathcal{B}_i$ be the collection of bounded sequences, that is $\Pi^*\mathcal{B}_i = \{f \in \Pi\mathcal{B}_i : \sup_{i \in I} \|x_i\|_i < \infty\}$. Then we can endow this space with a seminorm by taking the U-limit of the norms. To turn this into a norm, we must mod out by all of the sequences with 0 seminorm. That is, given $f, g \in \Pi^*\mathcal{B}_i$, set fU^*g iff $\lim_U \|f(i) - g(i)\|_i = 0$. Then $\Pi^*\mathcal{B}_i/U^*$ is a Banach space.

Chapter 3

Types of Infinite Tuples

3.1 Introduction

A well-known result in stability theory is that stability for 1-types implies stability for n-types for all $n < \omega$; see Shelah [Sh:c] Corollary I.§2.2 or Pillay [Pil83].0.9. In this chapter, we generalize this result to types of infinite length.

Theorem 3.1.1. *Given a complete theory T, if the supremum of the number of 1-types over models of size $\lambda \geq |T|$ is μ, then for any (possibly finite) cardinal $\kappa \leq \lambda$, the supremum of the number of κ-types over models of size λ is exactly μ^κ.*

We use our results to answer a question of Shelah from [Sh:c]. Rather than working in the context of first-order, we work with Galois types in Abstract Elementary Classes. This gives our results broader applicability. In particular, the above result holds for Galois types in AECs with λ-AP; see Theorem 3.3.1.

While the number of types of sequences of infinite lengths has not been calculated before, these types have already seen extensive use under the name tp_* in [Sh:c] and TP_* in [Sh:h].V.D.§3. While [Sh:c] uses them most extensively, it is the use in [Sh:h].V.D.§3 as types of models that might be most useful. This means that stability in λ can control the number of extensions of a model of size λ; see Section 3.3.

After seeing preliminary versions of this work, Rami Grossberg asked if the above theorem could be proved for nonalgebraic types. The examples in Proposition 3.4.1 show that such a theorem is not possible, even in natural elementary classes. However, we introduce a generalization of nonalgebraic types of tuples called strongly separative types for which we can prove the same upper bound. In AECs with disjoint amalgamation, such as elementary classes, nonalgebraic and strongly separative types coincide for types of length 1. For longer types, we require that realizations are, in a sense, nonalgebraic over each other. For instance, in ACF_0, the type of (e, π) can be considered "more nonalgebraic" over the set of algebraic numbers than the type of $(e, 2e)$. This is made precise in Definition 3.4.2.

Finally, in Section 3.5, we investigate the saturation of types of various lengths. The "saturation = model homogeneity" lemma (recall Lemma 2.1.16) shows that saturation is equivalent for all lengths. We also use bounds on the number of types and various structural properties to construct saturated models.

3.2 Preliminaries

We investigate the supremum of the number of types of a fixed length over all models of a fixed size. To simplify this discussion, we introduce the following notation.

Definition 3.2.1. *The* type bound for λ sized domains and κ lengths *is denoted* $\mathfrak{tb}_\lambda^\kappa = \sup_{M \in K_\lambda} |gS^\kappa(M)|$.

Shelah has introduced the notation of tp_* in [Sh:c].III.1.1 and TP_* in [Sh:h].V.D.3 to denote the types of infinite tuples, with tp_* having a syntactic definition (sets of formulas) and TP_* having a semantic definition (Galois types). Thus, $\mathfrak{tb}_\lambda^\kappa$ counts the maximum number of types of a fixed length κ over models of a fixed size λ, allowing for the possibility that this maximum is not achieved.

Clearly, λ-stability is the same as the statement that $\mathfrak{tb}_\lambda^1 = \lambda$. Also, we always have $\mathfrak{tb}_\lambda^1 \geq \lambda$ because each element in a model has a distinct type. Other notations have been used to count the supremum of the number of types, although the lengths have been finite. In [Kei76], Keisler uses

$$f_T(\kappa) = \sup\{|S^1(M, N)| : M, N \models T, M \prec N, \text{ and } \|M\| = \kappa\}$$

In [Sh:c].II.4.4, Shelah uses, for $\Delta \subset L(T)$ and $m < \omega$,

$$K_\Delta^m(\lambda, T) := \min\{\mu : |A| \leq \lambda \text{ implies } |S_\Delta^m(A)| < \mu\} = \sup_{|A|=\lambda} (|S_\Delta^m(A)|^+)$$

The relationships between these follow easily from the definitions

$$
\begin{aligned}
f_T(\kappa) &= \mathfrak{tb}_\lambda^1 \\
K_{L(T)}^m(\lambda, T) &= \sup_{\|M\|=\lambda} (|S^m(M)|^+) = \begin{cases} \mathfrak{tb}_\lambda^m & \text{if } K_{L(T)}^m(\lambda, T) \text{ is limit} \\ (\mathfrak{tb}_\lambda^m)^+ & \text{if } K_{L(T)}^m(\lambda, T) \text{ is successor} \end{cases} \\
&= \begin{cases} \mathfrak{tb}_\lambda^m & \text{if } \mathfrak{tb}_m^\lambda \text{ is a strict supremum} \\ (\mathfrak{tb}_\lambda^m)^+ & \text{if the supremum in } \mathfrak{tb}_m^\lambda \text{ is achieved} \end{cases}
\end{aligned}
$$

From this last equality, a basic question concerning $\mathfrak{tb}_\lambda^\kappa$ is if the supremum is strict or if there is a model that achieves the value. Below we describe two basic cases when the supremum in $\mathfrak{tb}_\lambda^\kappa$ is achieved.

Proposition 3.2.2. *Suppose K is an AEC with λ-AP and λ-JMP and $\kappa \leq \lambda$. If $cf\,\mathfrak{tb}_\lambda^\kappa \leq \lambda$ or if $I(K, \lambda) \leq \lambda$, then there is $M \in K_\lambda$ such that $|gS^\kappa(M)| = \mathfrak{tb}_\lambda^\kappa$.*

Proof: The idea of this proof is to put the $\leq \lambda$ many λ sized models together into a single λ sized model that will witness the conclusion. Pick $\langle M_i^* \in K_\lambda : i < \chi \rangle$ with $\chi \leq \lambda$ such that $\{|gS^\kappa(M_i^*)| : i < \chi\}$ has supremum $\mathfrak{tb}_\lambda^\kappa$; in the first case, this can be done by the definition of supremum and, in the second case, this can be done because there are only $I(K, \lambda)$ many possible values for $|gS^\kappa(M)|$ when $M \in K_\lambda$. Using amalgamation and joint mapping, we construct increasing and continuous $\langle N_i \in K_\lambda : i < \chi \rangle$ such that M_i^* is embeddable into N_{i+1}. Set $M = \cup_{i<\chi} N_i$. Since $\chi \leq \lambda$, we have $M \in K_\lambda$; this fact was also crucial in our construction. Since M_i^* can be embedded into M, we have that $|gS^\kappa(M_i^*)| \leq |gS^\kappa(M)| \leq \mathfrak{tb}_\lambda^\kappa$. Taking the supremum over all $i < \chi$, we get $\mathfrak{tb}_\lambda^\kappa = |gS^\kappa(M)|$, as desired. †

The use of joint embedding here seems necessary, at least from a naive point of view. It seems possible to have distinct AECs K^n in a common language that have models $M^n \in K_\lambda^n$ such that $|gS^\kappa(M^n)| = \mathfrak{tb}_\lambda^\kappa = \lambda^{+n}$, each computed in K^n. Then, we could form K^ω as the disjoint union of these classes; this would be an AEC with $\mathfrak{tb}_\lambda^\kappa = \lambda^{+\omega}$ and the supremum would not be achieved. However, examples of such K^n, even with $\kappa = 1$, are not known and the specified values of $|gS^\kappa(\cdot)|$ might not be possible.

These relationships help to shed light on a question of Shelah.

Question 3.2.3 ([Sh:c].III.7.6). *Is $K_{L(T)}^m(\lambda, T) = K_{L(T)}^1(\lambda, T)$ for $m < \omega$?*

The answer is yes, even for a more general question, under some cardinal arithmetic assumptions. Below, $\lambda^{(+\lambda^+)}$ denotes the λ^+th successor of λ^+.

Theorem 3.2.4. *Suppose $2^\lambda < \lambda^{(+\lambda^+)}$. If $\Delta \subset Fml(L(T))$ is such that $\phi(\mathbf{x}, x, \mathbf{y}) \in \Delta$ implies $\exists z \phi(\mathbf{x}, z, \mathbf{y}) \in \Delta$ and $n < \omega$, then*

$$K_\Delta^n(\lambda, T) = K_\Delta^1(\lambda, T)$$

Proof: There are two cases to consider: whether or not the supremum in $\mathfrak{tb}_\lambda^m$ is strict or is acheived. If the supremum is strict, then we claim the supremum in $\mathfrak{tb}_\lambda^1$ is strict as well. If not, there is some $M \in K_\lambda$ such that $|S^1(M)| = \mathfrak{tb}_\lambda^1$. But then, by Theorem 3.3.2,

$$\mathfrak{tb}_\lambda^m > |S^m(M)| \geq |S^1(M)|^m = (\mathfrak{tb}_\lambda^1)^m = \mathfrak{tb}_\lambda^m$$

a contradiction. So $\mathfrak{tb}_\lambda^m$ is a strict supremum and

$$K_{L(T)}^m(\lambda, T) = \mathfrak{tb}_\lambda^m = \mathfrak{tb}_\lambda^1 = K_{L(T)}^1(\lambda, T)$$

Note that this continues to hold if m is infinite or if we consider the corresponding relationship for Galois types in an AEC with amalgamation. Furthermore, this does not use the cardinal arithmetic assumption.

Now we consider the case that the supremum in $\mathfrak{tb}_\lambda^m$ is achieved and suppose for contradiction that the supremum in $\mathfrak{tb}_\lambda^1$ is strict. Then $m > 1$ and we assume it is the minimal such m. If $\mathfrak{tb}_\lambda^m = \mathfrak{tb}_\lambda^1$ is regular, than the pigeonhole argument used in Theorem 3.3.2 can find a model achieving $\mathfrak{tb}_\lambda^1$. In fact, this argument just requires that

$$\sup\{|S^{m-1}(Ma)| : a \vDash p, p \in S^1(M)\} < \lambda$$

By the remarks above the question, we know that $\operatorname{cf} \mathfrak{tb}_\lambda^1 > \lambda$ since the supremum is strict. This gives us that

$$\lambda < \operatorname{cf} \mathfrak{tb}_\lambda^1 < \mathfrak{tb}_\lambda^1 \leq 2^\lambda$$

However, this contradicts our cardinal arithmetic assumption because the minimal singular cardinal with cofinality above λ is $\lambda^{(+\lambda^+)} > 2^\lambda$. Thus

$$K_{L(T)}^m(\lambda, T) = (\mathfrak{tb}_\lambda^m)^+ = (\mathfrak{tb}_\lambda^1)^+ = K_{L(T)}^1(\lambda, T)$$

$$\dagger$$

3.3 Results on $S^\alpha(M)$

This section aims to prove Theorem 3.1.1 for AECs. In our notation, this can be stated as follows.

Theorem 3.3.1. *If K is an AEC with λ amalgamation, then for any $\kappa \leq \lambda$, allowing κ to be finite or infinite, we have $\mathfrak{tb}_\lambda^\kappa = (\mathfrak{tb}_\lambda^1)^\kappa$.*

We prove this by proving a lower bound (Theorem 3.3.2) and an upper bound (Theorem 3.3.4) for $\mathfrak{tb}_\lambda^\kappa$. Note that when $\kappa = \lambda$, this value is always the set-theoretic maximum, 2^λ. However, for $1 < \kappa < \min\{\chi : (\mathfrak{tb}_\lambda^1)^\chi = 2^\lambda\}$, this provides new information.

For readers interested in AECs beyond elementary classes, we note the use of amalgamation for the rest of this section and for the rest of this paper. It remains open whether these or other bounds can be found on the number of types without amalgamation. One possible obstacle is that different types cannot be put together: if we assume amalgamation, then given two types $p, q \in gS^1(M)$, there is some type $r \in gS^2(M)$ such that its first coordinate extends p and its second coordinate extends q. This will be a crucial tool in the proof of the lower bound. However, if we cannot amalgamate a model that realizes p and a model that realizes q over M, then such an extension type does not necessarily exist.

For the lower bound, we essentially "put together" all of the different types in $gS^1(M)$ as discussed above.

Theorem 3.3.2. *Let K be an AEC with λ-AP and λ-JMP. We have $\mathfrak{tb}_\lambda^\kappa \geq (\mathfrak{tb}_\lambda^1)^\kappa$. In particular, given $M \in K_\lambda$, $|gS^\kappa(M)| \geq |gS^1(M)|^\kappa$.*

Proof: We first prove the "in particular" clause and use that to prove the statement. Fix $M \in K_\lambda$ and set $\mu = |gS^1(M)|$. Fix some enumeration $\langle p_i : i < \mu \rangle$ of $gS^1(M)$. Then we claim that there is some $M^+ \succ M$ that realizes all of the types in $gS^1(M)$.

To see this, let $N_i \succ M$ of size λ contain a realization of p_i. Then set $M_0 = M$ and $M_1 = N_0$. For $\alpha = \beta + 1$, amalgamate M_β and N_β over M to get $M_\alpha \succ M_\beta$ and $f : N_\beta \to_M M_\alpha$; since N_β realizes $p_\beta \in S(M)$, $f(N_\beta)$ realizes $f(p_\beta) = p_\beta$. So M_α does as well. Take unions at limits. Then $M^+ := \cup_{\beta < \alpha} M_\beta$ realizes each type in $gS^1(M)$.

Having proved the claim, we show that $|gS^\kappa(M)| \geq \mu^\kappa$. For each $i < \mu$, pick $a_i \in |M^+|$ that realizes p_i. For each $f \in {}^\kappa\mu$, set $\mathbf{a}_f = \langle a_{f(i)} : i < \kappa \rangle$. We claim that the map $(f \in {}^\kappa\mu) \to gtp(\mathbf{a}_f/M, M^+)$ is injective, which completes the proof.

To prove injectivity, note that $gtp(a_j/M, M^+) = gtp(a_k/M, M^+)$ iff $j = k$. Suppose $gtp(\mathbf{a}_f/M, M^+) = gtp(\mathbf{a}_g/M, M^+)$. Then, we see that $gtp(a_{f(i)}/M, M^+) = gtp(a_{g(i)}/M, M^+)$ for each $i < \kappa$. By our above note, that means that $f(i) = g(i)$ for every $i \in \kappa = \text{dom } f = \text{dom } g$. So $f = g$. Thus, $|gS^\kappa(M)| \geq |{}^\kappa\mu| = \mu^\kappa$, as desired.

Now we prove that $\mathfrak{tb}_\lambda^\kappa \geq (\mathfrak{tb}_\lambda^1)^\kappa$. This is done by separating into cases based on $\text{cf}(\mathfrak{tb}_\lambda^1)$. If $\text{cf}(\mathfrak{tb}_\lambda^1) > \kappa$, then it is known that exponentiating to κ is continuous at $\mathfrak{tb}_\lambda^1$. Stated more plainly, if X is a set of cardinals such that $\text{cf}(\sup_{\chi \in X} \chi) > \kappa$, then

$$\left(\sup_{\chi \in X} \chi\right)^\kappa = \sup_{\chi \in X}(\chi^\kappa)$$

Then, we compute that

$$(\mathfrak{tb}_\lambda^1)^\kappa = (\sup_{M \in K_\lambda} |gS^1(M)|)^\kappa = \sup_{M \in K_\lambda}(|gS^1(M)|^\kappa) \leq \sup_{M \in K_\lambda} |gS^\kappa(M)| = \mathfrak{tb}_\lambda^\kappa$$

If $\text{cf}(\mathfrak{tb}_\lambda^1) \leq \kappa$, then we also have $\text{cf}(\mathfrak{tb}_\lambda^1) \leq \lambda$. By Proposition 3.2.2, we know that the supremum of $\mathfrak{tb}_\lambda^1$ is achieved, say by $M^* \in K_\lambda$. Then

$$(\mathfrak{tb}_\lambda^1)^\kappa = |gS^1(M^*)|^\kappa \leq |gS^\kappa(M^*)| \leq \sup_{M \in K_\lambda} |gS^\kappa(M)| = \mathfrak{tb}_\lambda^\kappa$$

$$\dagger$$

Now we show the upper bound. We do this in two steps. First, we present the "successor step" in Theorem 3.3.3 to give the reader the flavor of the argument. Then Theorem 3.3.4 gives the full argument using direct limits.

Theorem 3.3.3. *For any AEC K with λ-AP and any $n < \omega$, $\mathfrak{tb}_\lambda^n \leq \mathfrak{tb}_\lambda^1$.*

Note that, since it includes the $\|M\|$ many algebraic types, $gS^1(M)$ is always infinite, so this result could be written $\mathfrak{tb}_\lambda^n \leq (\mathfrak{tb}_\lambda^1)^n$.

Proof: We prove this by induction on $n < \omega$. The base case is $\mathfrak{tb}_\lambda^1 \leq \mathfrak{tb}_\lambda^1$. Suppose $\mathfrak{tb}_\lambda^n \leq \mathfrak{tb}_\lambda^1$ and set $\mu = \mathfrak{tb}_\lambda^1$. For contradiction, suppose there is some $M \in K_\lambda$ such that $|gS^{n+1}(M)| > \mu$. Then we can

find distinct $\{p_i \in S^{n+1}(M) \mid i < \mu^+\}$ and find $\langle a_j^i \mid j < n+1 \rangle \models p_i$ and $N_i \succ M$ that contains each a_j^i for $j < n+1$.

Consider $\{gtp(\langle a_j^i \mid j < n \rangle / M, N_i) : i < \mu^+\} \subset gS^n(M)$. By assumption, this set has size μ. So there is some $I \subset \mu^+$ of size μ^+ such that, for all $i \in I$, $gtp(\langle a_j^i \mid j < n \rangle / M, N_i)$ is constant.

Fix $i_0 \in I$. For any $i \in I$, the Galois types of $\langle a_j^i : j < n \rangle$ and $\langle a_j^{i_0} : j < n \rangle$ over M are equal. Thus, there are $N_i^* \succ N_{i_0}$ and $f_i : N_i \to_M N_i^*$ such that $f_i(a_j^i) = a_j^{i_0}$ for all $j < n$ and

$$
\begin{array}{ccc}
N_{i_0} & \longrightarrow & N_i^* \\
\uparrow & & \uparrow {\scriptstyle f_i} \\
M & \longrightarrow & N_i
\end{array}
$$

commutes. Now consider the set $\{gtp(f_i(a_n^i)/N_{i_0}, N_i^*) \mid i \in I\}$. We have that $|I| = \mu^+$ and $|S^1(N_{i_0})| \leq \mathfrak{tb}_\lambda^1 = \mu$, so there is $I^* \subset I$ of size μ^+ so, for all $i \in I^*$, $gtp(f_i(a_n^i)/N_{i_0}, N_i^*)$ is constant. Let $i \neq k \in I^*$. Then $gtp(f_i(a_n^i)/N_{i_0}, N_i^*) = gtp(f_k(a_n^k)/N_{i_0}, N_k^*)$. By the definition of Galois types, we can find N^{**}, $g_k : N_k^* \to N^{**}$, and $g_i : N_i^* \to N^{**}$ such that $g_k(f_k(a_n^k)) = g_i(f_i(a_n^i))$ and the following commutes

$$
\begin{array}{ccc}
N_k^* & \xrightarrow{\ g_k\ } & N^{**} \\
\uparrow & & \uparrow {\scriptstyle g_i} \\
N_{i_0} & \longrightarrow & N_i^*
\end{array}
$$

We put these diagrams together and get the following:

Thus, we have amalgamated N_i^* and N_k^* over M. Furthermore, for each $j < n+1$, we have $g_k(f_k(a_j^k)) = g_i(f_i(a_j^i))$. This witnesses $gtp(\langle a_j^i \mid j < n+1 \rangle / M, N_i) = gtp(\langle a_j^j \mid j < n+1 \rangle / M, N_j)$, which is a contradiction.

Thus, $|gS^{n+1}(M)| \leq \mu = \mathfrak{tb}_1^\lambda$ for all $M \in K_\lambda$ as desired. $\qquad\qquad$ †

This proof can be seen as a semantic generalization of the proof that stability for 1-types implies stability. Now we wish to prove this upper bound for types of any length $\leq \lambda$.

The proof works by induction to construct a tree of objects that is indexed by $(\mathfrak{tb}_\lambda^1)$–called μ in the proof–that codes all κ length types as its branches. Successor stages of the construction are similar to the above proof, but with added bookkeeping. At limit stages, we wish to continue the construction in a continuous way. However, we will have a family of embeddings rather than an increasing $\prec_K$-chain. This is fine since closure under direct limits follows from the AEC axioms; recall Fact 2.1.5.

We now prove the main theorem.

Theorem 3.3.4. *If K is an AEC with λ-AP and $\kappa \leq \lambda$, then $\mathfrak{tb}_\lambda^\kappa \leq (\mathfrak{tb}_\lambda^1)^\kappa$.*

Proof: Set $\mu = \mathfrak{tb}_\lambda^1$. Let $M \in K_\lambda$ and enumerate $gS^\kappa(M)$ as $\langle p_i \in gS^\kappa(M) : i < \chi \rangle$, where $\chi = |gS^\kappa(M)|$. We will show that $\chi \leq \mu^\kappa$, which gives the result. For each $i < \chi$, find $N_0^i \in K_\lambda$ such that $M \prec N_0^i$ and there is $\langle a_i^\alpha \in |N_0^i| : \alpha < \kappa \rangle \models p_i$.

The formal construction is laid out below, but we give the idea first. Our construction will essentially create three objects: a tree of models $\langle M_\eta : \eta \in {}^{<\kappa}\mu \rangle$; for each $i < \chi$, a function $\eta_i : \kappa \to \mu$; and, for each $i < \chi$, a coherent, continuous system $\{N_\alpha^i, \widehat{f}_{\beta,\alpha}^i : \beta < \alpha < \kappa\}$. The tree of models will be domains of types such that the relation of M_η to $M_{\eta^\frown j}$ is like that of M to N_{i_0} in Theorem 3.3.3. We would like the value of the function η_i at some $\alpha < \kappa$ to determine the type of a_i^α over $M_{\eta_i \restriction \alpha}$. This can't work because a_i^α isn't in a model also containing M_ν; instead we use its image $\widehat{f}_{0,\alpha+1}^i(a_i^\alpha)$ under the coherent system. At successor stages of our construction, we will put together elements of equal type over a fixed witness (i_η here standing in for i_0 in Theorem 3.3.3). At limit stages, we take direct limits.

Once we finish our construction, we show that the map $i \in \chi \mapsto \eta_i \in {}^\kappa\mu$ is injective. This is done by putting the type realizing sequence together along the chain $\langle M_{\eta_i \restriction \alpha} : \alpha < \kappa \rangle$ to show that η_i characterizes p_i.

More formally, we construct the following:

1. A continuous tree of models $\langle M_\eta \in K_\lambda : \eta \in {}^{<\kappa}\mu \rangle$ with an enumeration of the types over each model $gS^1(M_\eta) = \{p_j^\eta : j < |gS^1(M_\eta)|\}$.

2. For each $i < \chi$, a function $\eta_i \in {}^\kappa\mu$.

3. For each $\eta \in {}^{<\kappa}\mu$, an ordinal $i_\eta < \chi$.

4. For each $i < \chi$, a coherent, continuous system $\{N_\alpha^i, \widehat{f}_{\beta,\alpha}^i : N_\beta^i \to_{M_{\eta_i \restriction \beta}} N_\alpha^i : \beta < \alpha < \kappa\}$; that is, one such that $\gamma < \beta < \alpha < \kappa$ implies $\widehat{f}_{\gamma,\alpha}^i = \widehat{f}_{\beta,\alpha}^i \circ \widehat{f}_{\gamma,\beta}^i$ and so $\delta < \kappa$ limit implies $(N_\delta^i, \widehat{f}_{\alpha,\delta}^i)_{\alpha<\delta} = \varinjlim_{\gamma<\beta<\delta}(N_\alpha^i, \widehat{f}_{\gamma,\beta}^i)$.

Our construction will have the following properties for all $\eta \in {}^\beta\mu$ when $\beta < \kappa$.

(A) $i_\eta = \min\{i < \chi : \eta < \eta_i\}$ if that set is nonempty.

(B) $M_{\eta^\frown \langle j \rangle} := N_\beta^{i_{\eta^\frown \langle j \rangle}}$ and $M_{\eta_i \restriction \beta} \prec N_\beta^i$.

(C) If $\eta^\frown \langle j \rangle < \eta_i$, then $p_j^\eta = gtp(\widehat{f}_{0,\beta}^i(a_i^\beta)/M_\eta, N_\beta^i)$. In particular, this is witnessed by the following diagram

$$
\begin{array}{ccc}
N_\beta^{i_\eta} & \longrightarrow & N_{\beta+1}^i \\
\uparrow & & \uparrow \scriptstyle{\widehat{f}_{\beta,\beta+1}^i} \\
M_\nu & \longrightarrow & N_\beta^i
\end{array}
$$

with $\widehat{f}_{0,\beta+1}^i(a_i^\beta) = \widehat{f}_{0,\beta}^{i_{\eta^\frown \langle j \rangle}}(a_{i_{\eta^\frown \langle j \rangle}}^\beta)$

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Construction: At stage $\alpha < \kappa$ of the construction, we will construct $\langle M_\eta : \eta \in {}^\alpha\mu\rangle$, $\eta_i \upharpoonright \alpha$, and $\{N_\alpha^i, \widehat{f}_{\beta,\alpha}^i : \beta < \alpha\}$ for all $i < \chi$.

$\underline{\alpha = \emptyset}$: We set $M_\emptyset = M$ and note that N_0^i is already defined. Then $\widehat{f}_{0,0}^i$ is the identity.

$\underline{\alpha \text{ is limit}}$: For each $\eta \in {}^\alpha\mu$, set $M_\eta = \cup_{\beta<\alpha} M_{\eta\upharpoonright\beta}$ and $(N_\alpha^i, \widehat{f}_{\beta,\alpha}^i)_{\alpha<\delta} = \varinjlim_{\gamma<\beta<\delta}(N_\alpha^i, \widehat{f}_{\gamma,\beta}^i)$ as required. The values of $\eta_i \upharpoonright \alpha$ are already determined by the earlier phases of the construction.

$\underline{\alpha = \beta+1}$: We have constructed our system for each $\nu \in {}^\beta\mu$. This means that there are enumerations $\{p_k^\nu : k < |gS^1(M_\nu)|\}$ of the 1-types with domain M_ν. Then, if $i < \chi$ such that $\nu = \eta_i \upharpoonright \beta$, we set

$$\eta_i(\beta) = k, \text{ where } k < \mu \text{ is unique such that } gtp(\widehat{f}_{0,\beta}^i(a_i^\beta)/M_\nu, N_\beta^i) = p_k^\nu$$

Then, for each $\eta \in {}^\alpha\mu$ set $i_\eta = \min\{i < \chi : \eta_i \upharpoonright \alpha = \eta\}$ if this set is nonempty; pick it arbitrarily otherwise. Then, for all $i < \chi$, we have that

$$gtp(\widehat{f}_{0,\beta}^i(a_i^\beta)/M_\nu, N_\beta^i) = gtp(\widehat{f}_{0,\beta}^{i_{\eta_i\upharpoonright\alpha}}(a_{i_{\eta_i\upharpoonright\alpha}}^\beta)/M_\nu, N_\beta^{i_{\eta_i\upharpoonright\alpha}})$$

This Galois type equality means that there is a model $N_{\beta+1}^i \succ N_\beta^{i_{\eta_i\upharpoonright\alpha}}$ and a function $\widehat{f}_{\beta,\beta+1}^i : N_\beta^i \to_{M_\nu} N_{\beta+1}^i$ such that

$$\widehat{f}_{\beta,\beta+1}^i(\widehat{f}_{0,\beta}^i(a_i^\beta)) = \widehat{f}_{0,\beta}^{i_{\eta_i\upharpoonright\alpha}}(a_{i_{\eta_i\upharpoonright\alpha}}^\beta)$$

Set $M_\eta = N_\beta^{\eta_i\upharpoonright\alpha}$ (note that this doesn't depend on the choice of i) and, for $\gamma \leq \beta$, set $\widehat{f}_{\gamma,\beta+1}^i = \widehat{f}_{\beta,\beta+1}^i \circ \widehat{f}_{\gamma,\beta}^i$. This completes the construction.

This is enough: As indicated above, we will show that the map from i to η_i is injective. We do this by showing that $\eta_i = \eta_j$ implies $p_i = p_j$ and, recalling that the enumeration of the p_i were distinct, we must have $i = j$.

Let $i, j < \chi$ such that $\eta := \eta_i = \eta_j$. We want to show $p_i = p_j$. We have the following commuting diagram of models for each $\beta < \alpha < \kappa$:

$$
\begin{array}{ccccc}
N_0^j & \longleftarrow & M_{\eta\upharpoonright 0} & \longrightarrow & N_0^i \\
\widehat{f}_{0,\beta}^j \downarrow & & \downarrow & & \downarrow \widehat{f}_{0,\beta}^i \\
N_\beta^j & \longleftarrow & M_{\eta\upharpoonright\beta} & \longrightarrow & N_\beta^i \\
\widehat{f}_{\beta,\alpha}^j \downarrow & & \downarrow & & \downarrow \widehat{f}_{\beta,\alpha}^i \\
N_\alpha^j & \longleftarrow & M_{\eta\upharpoonright\alpha} & \longrightarrow & N_\alpha^i
\end{array}
$$

with the property that, for each $\alpha < \kappa$, we know

$$
\begin{aligned}
\widehat{f}_{0,\alpha+1}^i(a_i^\alpha) &= \widehat{f}_{0,\alpha}^{i_{\eta\upharpoonright\alpha+1}}(a_{i_{\eta\upharpoonright\alpha+1}}^\alpha) \\
&= \widehat{f}_{0,\alpha+1}^j(a_j^\alpha)
\end{aligned}
$$

Note that this element is in $M_{\eta\upharpoonright\alpha+1}$. Now set $\widehat{M} = \cup_{\alpha<\kappa} M_{\eta\upharpoonright\alpha}$.

Let k stand in for either i or j. Set $(\widehat{N}^k, \widehat{f}_{\alpha,\infty}^k)_{\alpha<\kappa} = \varinjlim_{\gamma<\beta<\kappa}(N_\beta^k, \widehat{f}_{\gamma,\beta}^k)$. This gives us the following diagram.

$$
\begin{array}{ccccc}
N_0^i & \longleftarrow & M & \longrightarrow & N_0^j \\
\widehat{f}_{0,\infty}^i \downarrow & & \downarrow & & \downarrow \widehat{f}_{0,\infty}^j \\
\widehat{N}^i & \longleftarrow & \widehat{M} & \longrightarrow & \widehat{N}^j
\end{array}
$$

Then we can amalgamate $\widehat{N}^j$ and $\widehat{N}^i$ over $\widehat{M}$ with

$$
\begin{array}{ccc}
\widehat{N}^j & \xrightarrow{\;g\;} & N^* \\
\uparrow & & \uparrow{\scriptstyle f} \\
\widehat{M} & \longrightarrow & \widehat{N}^i
\end{array}
$$

Then, for all $\alpha < \kappa$ and $k = i, j$, $\widehat{f}^k_{0,\infty}(a^\alpha_k) = \widehat{f}^k_{\alpha+1,\infty}(\widehat{f}^k_{0,\alpha+1}(a^\alpha_k))$. We know that $\widehat{f}^k_{0,\alpha+1}(a^\alpha_k) \in |M_{\eta\restriction\alpha+1}|$, so it is fixed by f^k_β for $\beta > \alpha + 1$. This means it is also fixed by $\widehat{f}^k_{\alpha+1,\infty}$. Then

$$
\widehat{f}^k_{0,\infty}(a^\alpha_k) = \widehat{f}^k_{\alpha+1,\infty}(\widehat{f}^k_{0,\alpha+1}(a^\alpha_k)) = \widehat{f}^k_{0,\alpha+1}(a^\alpha_k) = \widehat{f}^{i_{\eta\restriction\alpha+1}}_{0,\alpha}(a^\alpha_{i_{\eta\restriction\alpha+1}})
$$

Since this last term is independent of whether k is i or j, we have $\widehat{f}^i_{0,\infty}(a^\alpha_i) = \widehat{f}^j_{0,\infty}(a^\alpha_j) \in \widehat{M}$ for all $\alpha < \kappa$. Since our amalgamating diagram commutes over $\widehat{M}$, $f(\widehat{f}^i_0(a^\alpha_i)) = g(\widehat{f}^j_0(a^\alpha_j))$.
Combining the above, we have

$$
\begin{array}{ccc}
N^j_0 & \xrightarrow{\;g\circ\widehat{f}^j_{0,\infty}\;} & N^* \\
\uparrow & & \uparrow{\scriptstyle f\circ\widehat{f}^i_{0,\infty}} \\
M & \longrightarrow & N^i_0
\end{array}
$$

with $f \circ \widehat{f}^i_{0,\infty}(\langle a^\alpha_i \mid \alpha < \kappa\rangle) = g \circ \widehat{f}^j_{0,\infty}(\langle a^\alpha_j \mid \alpha < \kappa\rangle)$.
Thus,

$$
p_i = gtp(\langle a^\alpha_i \mid \alpha < \kappa\rangle/M, N^i_0) = gtp(\langle a^\alpha_j \mid \alpha < \kappa\rangle/M, N^j_0) = p_j
$$

Since each p_k was distinct, this implies that $i = j$. The map $i \mapsto \eta_i$ is injective and $\chi \leq \mu^\kappa$ as desired. †

As mentioned in the Introduction, the above result gives us the proof of Theorem 3.1.1:

Proof of Theorem 3.1.1: As discussed in the last section, $(\mathrm{Mod}\ T, \prec_{L(T)})$ is an AEC with amalgamation over sets. Given a set A, passing to a model containing A can only increase the number of types. Thus, even in this case, it is enough to only consider models when computing $\mathfrak{tb}$. Thus,

$$
\sup_{A\subset M\models T, \|A\|=\lambda} |S^\mu(A)| = \mathfrak{tb}^\mu_\lambda = (\mathfrak{tb}^1_\lambda)^\mu = \left(\sup_{A\subset M\models T, \|A\|=\lambda} |S^1(A)|\right)^\mu
$$

as desired. †

After seeing this work, Alexei Kolesnikov pointed out a much simpler proof of Theorem 3.3.4 for first order theories or, more generally, for AECs that are $< \omega$ type short over λ-sized domains; in either case, a type of infinite length is determined by its restrictions to finite sets of variables. Fix a type $p \in S^I(M)$ with I infinite. The previous comment means that the map

$$
p \mapsto \Pi_{\mathbf{x}\in[I]^{<\omega}} p^{\mathbf{x}}
$$

from $S^I(M)$ to $\Pi_{\mathbf{x}\in[I]^{<\omega}} S^{\mathbf{x}}(M)$ is injective. Then

$$
\begin{aligned}
|S^I(M)| &\leq \Pi_{\mathbf{x}\in[I]^{<\omega}}|S^{\mathbf{x}}(M)| = \Pi_{\mathbf{x}\in[I]^{<\omega}}|S^1(M)| \\
&= |S^1(M)|^{|[I]^{<\omega}|} = |S^1(M)|^{|I|}
\end{aligned}
$$

This is in fact a strengthening of Theorem 3.3.4 as in Theorem 3.3.2.

We now examine local types in first order theories. For $\Delta \subset Fml(L(T))$, set

$$\Delta\mathfrak{tb}^\lambda_\kappa = \sup_{M \models T, \|M\| = \lambda} |S^\kappa_\Delta(M)|$$

If $\Delta = \{\phi\}$, we simply write $\phi\mathfrak{tb}^\kappa_\lambda$. Unfortunately, there is no semantic equivalent of Δ-types, so the methods and proofs above do not transfer. For a lower bound, we can prove the following in the same way as Theorem 3.3.2.

Proposition 3.3.5. *If T is a first order theory and $\Delta \subset Fml(L(T))$, then for any κ we have that $|S^1_\Delta(A)| = \mu$ implies that $|S^\kappa_\Delta(A)| \geq \mu^\kappa$.*

If Δ is closed under existential quantification, the syntactic proofs of Theorem 3.3.3 (see, for instance, [Sh:c].2) can be used to get an upper bound for $\Delta\mathfrak{tb}^n_\lambda$ when n is finite.

Proposition 3.3.6. *If for all $\phi(\mathbf{x}, x, \mathbf{y}) \in \Delta$, we have $\exists z \phi(\mathbf{x}, z, \mathbf{y}) \in \Delta$, then $\Delta\mathfrak{tb}^n_\lambda \leq \Delta\mathfrak{tb}^1_\lambda$ for $n < \omega$.*

With this result for finite lengths, we can apply the syntactic argument above to conclude the following.

Proposition 3.3.7. *If T is a first order theory and $\Delta \subset Fml(L(T))$, then for $\kappa \leq \lambda$,*

$$\Delta\mathfrak{tb}^\kappa_\lambda \leq (\sup_{n < \omega} \Delta\mathfrak{tb}^n_\lambda)^\kappa$$

In particular, if Δ is closed under existentials as in Proposition 3.3.6, then $\Delta\mathfrak{tb}^\kappa_\lambda \leq (\Delta\mathfrak{tb}^1_\lambda)^\kappa$.

We now turn to the values of $\phi\mathfrak{tb}$ for particular ϕ. Recall that Theorem [Sh:c].II.2.2 says that T is stable iff T is λ stable for $\lambda = \lambda^{|T|}$ iff it is λ stable for ϕ types for all $\phi \in L(T)$. This means that if T is unstable in $\lambda = \lambda^{|T|}$, then there is some ϕ such that $\phi\mathfrak{tb}^1_\lambda > \lambda$. Further, suppose that $\sup\{\psi\mathfrak{tb}^1_\lambda : \psi \in L(T)\} = \lambda^{+n}$ for some $1 \leq n < \omega$. Then, since λ^{+n} is a successor, this supremum is acheived by some formula ϕ_λ. Then, since $\lambda^{|T|} = \lambda$, we can calculate

$$\begin{aligned}
\phi_\lambda\mathfrak{tb}^1_\lambda &= \sup_{\psi \in L(T)} \{\psi\mathfrak{tb}^1_\lambda\} \leq \mathfrak{tb}^1_\lambda \leq \Pi_{\psi \in L(T)}(\psi\mathfrak{tb}^1_\lambda) \leq (\phi_\lambda\mathfrak{tb}^1_\lambda)^{|T|} = \\
&= (\lambda^{+n})^{|T|} = \lambda^{|T|} \cdot \lambda^{+n} = \lambda^{+n} = \phi_\lambda\mathfrak{tb}^1_\lambda
\end{aligned}$$

So $\phi_\lambda\mathfrak{tb}^1_\lambda = \mathfrak{tb}^1_\lambda$. Thus, for all $\kappa \leq \lambda$, we can use Theorems 3.3.5 and 3.3.4 to calculate

$$(\phi_\lambda\mathfrak{tb}^1_\lambda)^\kappa \leq \phi_\lambda\mathfrak{tb}^\kappa_\lambda \leq \mathfrak{tb}^\kappa_\lambda = (\mathfrak{tb}^1_\lambda)^\kappa = (\phi_\lambda\mathfrak{tb}^1_\lambda)^\kappa$$

This gives us the following result:

Theorem 3.3.8. *Given a first order theory T, if λ is a cardinal such that $\lambda^{|T|} = \lambda$ and $\sup\{|S^1_\psi(A)| : \psi \in L(T), |A| \leq \lambda\} < \lambda^{+\omega}$, then there is some $\phi_\lambda \in L(T)$ such that, for all $\kappa \leq \lambda$, $\mathfrak{tb}^\kappa_\lambda = \phi_\lambda\mathfrak{tb}^\kappa_\lambda$.*

Returning to general AECs, in [Sh:h].V.D.§3, Shelah considers long types of tuples enumerating a model extending the domain. In this case, any realization of the type is another model extending the domain that is isomorphic to the original tuple over the domain. Thus, an upper bound on types of a certain length κ also bounds the number of isomorphism classes extending the domain by κ many elements. More formally, we get

Proposition 3.3.9. *Given $M \in K_\lambda$,*

$$|\{N/\cong_M \colon N \in K, M \subsetneqq N, |N - M| = \kappa\}| \leq \mathfrak{tb}_\lambda^\kappa$$

If we have an AEC with amalgamation where any extension can be broken into smaller extensions, this could lead to a useful analysis. Unfortunately, this provides us with no new information when $\kappa = \lambda$ since $2^\lambda = \mathfrak{tb}_\lambda^\lambda$ is already the well-known upper bound for λ-sized extensions of M and there are even first order theories where $M \subsetneqq N$ implies $|N - M| \geq \|M\|$. Algebraically closed fields of characteristic 0 are such an example.

3.4 Strongly Separative Types

One might hope that similar bounds could be developed for non-algebraic types. This would probably give us a finer picture of what is going on because a model M necessarily has at least $\|M\|$ many algebraic types over M, so in the stable case, the number of non-algebraic types could, *a priori*, be anywhere between 0 and $\|M\|$; the case $gS_{na}^1(M) = \emptyset$ only occurs in the uninteresting case that M has no extensions.

However, as the following result shows, no such result is possible even in basic, well-understood first order cases:

Proposition 3.4.1.   *1. Let T_1 be the empty theory and $M \vDash T_1$. Then $|S_{na}^1(M)| = 1$ and $|S_{na}^n(M)| = B_n$ for all $n < \omega$, where B_n is the nth Bell number. In particular, this is finite.*

  2. Let $T_2 = ACF_0$ and $M \vDash T$. Then $|S_{na}^1(M)| = 1$ but $|S_{na}^2(M)| = \|M\|$.

Note that these examples represent the minimal and maximal, respectively, number of long, nonalgebraic types given that there is only one non-algebraic type.

Proof:

1. Let $tp(a/M, N_1), tp(b/M, N_2) \in S_{na}^1(M)$ and, WLOG, assume $\|N_1\| \leq \|N_2\|$. Then let f fix M, send a to b, and injectively map $N_1 - M - \{a\}$ to $N_2 - M - \{b\}$ arbitrarily. This witenesses $tp(a/M, N_1) = tp(b/M, N_2)$.
   Given the type of $\langle a_n : n < k \rangle$, the only restriction on finding a function to witness type equality is given by which elements of the sequence are repeated; for instance, if $a \neq b$, then $tp(a, b/M, N) \neq tp(a, a/M, N)$. Thus, each type can be represented by those elements of the sequence which are repeated. To count this, we need to know the number of partitions of n. This is given by Bell's numbers, defined by $B_1 = 1$ and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. See [Wil94].1.6.13 for a reference. Then, the number of n-types is just B_n.

2. This is an easy consequence of Steinitz's Theorem that there is only one non-algebraic 1-type, that of an element transcedental over the domain.
   Looking at 2-types, let $M \in K$ and $e \in N \succ M$ be transcendental (non-algebraic) over M. For each polynomial $f \in M[x]$, set $p_f = tp(e, f(e)/M, N)$. Then, for $f \neq g$, we have that $p_f \neq p_g \in S_{na}^2(M)$. Also, $tp(e, \pi/M, N')$ is distinct, where π is transcendental over $M(e)$. This gives at least $\|M\|$ many 2-types.
   We know there are at most $\|M\|$ many because the theory is stable. Therefore, the results of last

section tells us that there are exactly $\|M\|^2 = \|M\|$ many 2-types, so there are at most $\|M\|$ many non-algebraic 2-types. †

This shows that a result like Theorem 3.3.4 is impossible for non-algebraic types. As is evident in the proof above, especially part two, the variance in the number of types comes from the fact that, while the realizations of the non-algebraic type are not algebraic over the model, they might be algebraic over each other. This means that even 2-types, like $tp(e, 2e/\mathbb{A}, \mathbb{C})$, that are not realized in the base model can't be separated: any algebraically closed field realizing the type of e must also realize the type of $2e$.

In order to get a bound on the number of these types, we want to be able to *separate* the different elements of the tuples that realize the long types. This motivates our definition and naming of separative types below. We also introduce a slightly stronger notion, strongly separative types, that allow us to not only separate realizations of the type, but also gives us the ability to extend types, as made evident in Proposition 3.4.6. Luckily, in the first order case and others, these two notions coincide; see Proposition 3.4.5.

Definition 3.4.2.

1. *We say that a triple $(\langle a_i : i < \alpha \rangle, M, N) \in K_\lambda^{3,\alpha}$ is* separative *iff there are increasing sequences of intermediate models $\langle N_i \in K : i < \alpha \rangle$ such that, for all $i < \alpha$, $M \prec N_i \prec N$ and $a_i \in N_{i+1} - N_i$. The sequence $\langle N_i : i < \alpha \rangle$ is said to witness the triple's separativity.*

2. *For $M \in K$, set $gS_{sep}^\alpha(M) = \{gtp(\langle a_i : i < \alpha, \rangle / M, N) : (\langle a_i : i < \alpha, \rangle, M, N) \in K_\lambda^{3,\alpha}$ is separative$\}$.*

3. *We say that a triple $(\langle a_i : i < \alpha \rangle, M, N) \in K_\lambda^{3,\alpha}$ is* strongly separative *iff there is a sequence witnessing its separativity $\langle N_i : i < \alpha \rangle$ that further has the property that, for any $i < \alpha$ and $N_1^+ \succ N_i$ of size λ, there is some $N_2^+ \succ N_1^+$ and $g : N_{i+1} \to_{N_\beta} N_2^+$ such that $g(a_i) \notin N_1^+$.*

4. *For $M \in K_\lambda$, set $gS_{strsep}^\alpha(M) = \{gtp(\langle a_i : i < \alpha, \rangle / M, N) : (\langle a_i : i < \alpha, \rangle, M, N) \in K_\lambda^{3,\alpha}$ is strongly separative$\}$.*

The condition "$a_i \in N_{i+1} - N_i$" in (1) could be equivalently stated as either of the following:

- For all $j < \alpha$, $a_j \in N_i$ iff $i < j$.

- $gtp(a_i/N_i, N_{i+1})$ is nonalgebraic.

Note that the examples in Proposition 3.4.1 only have one separative or strongly separative type of any length: for the empty theory, this is any sequence of distinct elements and, for ACF_0, this is any sequence of mutually transcendental elements. Theorem 3.4.8 below shows this generally by proving the upper bound from the last section (Theorem 3.3.4) holds for strongly separative types. Before this proof, a few comments about these definitions are in order.

First, the key part of the definition is about triples, but we will prove things about types. This is not an issue because any triple realizing a (strongly) separative type can be made into a (strongly) separative type by extending the ambient model.

Proposition 3.4.3.

1. *If $gtp(\langle a_\beta : \beta < \alpha \rangle / M, N) \in gS^\alpha_{sep}(M)$, then there is some $N^+ \succ N$ such that $(\langle a_\beta : \beta < \alpha \rangle, M, N^+)$ is separative.*

2. *The same is true for strongly separative types.*

Proof: We will prove the first assertion and the second one follows similarly. By the definition of gS^α_{sep}, there is some separative $(\langle b_\beta : \beta < \alpha \rangle, M, N_1) \in K^{3,\alpha}_\lambda$ such that $gtp(\langle a_\beta : \beta < \alpha \rangle / M, N) = gtp(\langle b_\beta : \beta < \alpha \rangle / M, N)$. Thus, there exists some $N^+ \succ N$ and $f : N_1 \rightarrow_M N^+$ such that $f(b_\beta) = a_\beta$ for all $\beta < \alpha$. Let $\langle N_\beta : \beta < \alpha \rangle$ be a witness sequence to $(\langle b_\beta : \beta < \alpha \rangle, M, N_1)$'s separativity. Then $\langle f(N_\beta) \prec N^+ : \beta < \alpha \rangle$ is a witness sequence for $(\langle a_\beta : \beta < \alpha \rangle, M, N^+)$. †

Second, although we continue to use the semantic notion of types (Galois types) for full generality, these notions are new in the context of first order theories. In this context, the elements of the witnessing sequence $\langle N_i : i < \alpha \rangle$ are still required to be models, even though types are meaningful over sets. An attempt to characterize these definitions in a purely syntactical nature (i.e. by only mentioning formulas) was unsuccessful, but we do know (see Proposition 3.4.5 below) that all separative types over models are strongly separative for complete first order theories.

Third, we can easily characterize these properties for 1-types.

Proposition 3.4.4. *Let K be an AEC and $p \in gS^1(M)$.*

- *p is separative iff p is nonalgebraic.*

- *p is strongly separative iff, for any $N \succ M$ with $\|N\| = \|M\|$, there is an extension of p to a non-algebraic type over N. Such types are called* big.

Finally, strongly separative types and separative types are the same in the presence of the disjoint amalgamation property.

Proposition 3.4.5. *Let α be an ordinal and $M \in K$. If K satisfies the disjoint amalgamation property when all models involved have sizes between $\|M\|$ and $|\alpha| + \|M\|$, inclusive, then $gS^\alpha_{strsep}(M) = gS^\alpha_{sep}(M)$.*

Proof: By definition, $gS^\alpha_{strsep}(M) \subset gS^\alpha_{sep}(M)$, so we wish to show the other containment. Let $gtp(\langle a_\beta : \beta < \alpha \rangle / M, N) \in gS^\alpha_{sep}(M)$. Let $\langle N_\beta : \beta < \alpha \rangle$ be a witnessing sequence and let $N_1^+ \succ N_{\beta_0}$ of size $\|N_{\beta_0}\|$ for some $\beta_0 < \alpha$. By renaming elements, we can find some copy of N_1^+ that is disjoint from N_{β_0+1} except for N_{β_0}. So there are $\widehat{N}$ and $f : N_1^+ \cong_{N_{\beta_0}} \widehat{N}$ such that $\widehat{N} \cap N_{\beta_0+1} = N_{\beta_0}$. Then, we can use disjoint amalgamation on $\widehat{N}$ and N_{β_0+1} over N_{β_0} to get N^* and $g : N_{\beta_0+1} \rightarrow N^*$ so

commutes and $\widehat{N} \cap g(N_{\beta_0+1}) = N_{\beta_0}$. Thus, since a_{β_0} is in N_{β_0+1} and not in N_{β_0}, we have that $g(a_{\beta_0})$ is in $g(N_{\beta_0+1})$ and not in $\widehat{N}$. Let $\widehat{f}$ be an $L(K)$-isomorphism that extends f and has N^* in its range. Then we have

$$\widehat{f}^{-1}(g(a_{\beta_0})) \notin \widehat{f}^{-1}(\widehat{N}) = f^{-1}(\widehat{N}) = N_1^+$$

Then we can collapse the above diagram to

$$
\begin{array}{ccc}
N_1^+ & \longrightarrow & \widehat{f}^{-1}(N^*) \\
\uparrow & & \uparrow {\scriptstyle \widehat{f}^{-1} \circ g} \\
N_{\beta_0} & \longrightarrow & N_{\beta_0+1}
\end{array}
$$

This diagram commutes and witnesses the property for strong separativity with $N_2^+ = \widehat{f}^{-1}(N^*)$. †

It is an exercise in the use of compactness that every complete first order theory satisfies disjoint amalgamation over models, see Hodges [Hod93].6.4.3 for a reference. For a general AEC, this is not the case. Baldwin, Kolesnikov, and Shelah [BKS927] have constructed examples of AECs without disjoint amalgamation. On the other hand, Shelah [Sh576] has shown that disjoint amalgamation follows from certain amounts of structure (see, in particular, 2.17 and 5.11 there). Additionally, Grossberg, VanDieren, and Villaveces [GVV] point out that many AECs with a well developed independence notion, such as homogeneous model theory or finitary AECs, also satisfy disjoint amalgamation.

In order to prove the main theorem of this section, Theorem 3.4.8, we will need to make use of certain closure properties of strongly separative types. These also hold for separative types as well.

Proposition 3.4.6 (Closure of gS_{strsep}).

1. *If $p \in gS_{strsep}^\alpha(M)$ and $I \subset \alpha$, then $p^I \in gS_{strsep}^{otp(I)}(M)$.*

2. *If $p \in gS_{strsep}^\alpha(M)$ and $M_0 \prec M$, then $p \restriction M_0 \in gS_{strsep}^\alpha(M_0)$.*

We now prove the main theorem.

Definition 3.4.7. *The* strongly separative type bound for λ sized domains and κ lengths *is denoted* $strsep\mathfrak{tb}_\lambda^\kappa = \sup_{M \in K_\lambda} |gS_{strsep}^\kappa(M)|$.

Theorem 3.4.8. *If $strsep\mathfrak{tb}_\lambda^1 = \mu$, then $strsep\mathfrak{tb}_\lambda^\kappa \le \mu^\kappa$ for all (possibly finite) $\kappa \le \lambda^+$.*

Proof: The proof is very similar to that of Theorem 3.3.4, so we only highlight the differences.

As before, let $M \in K_\lambda$, enumerate $gS_{strsep}^\kappa(M) = \langle p_i : i < \chi \rangle$, and find $N_0^i \succ M$ of size $\lambda + \kappa$ and $a_i^\alpha \in |N_0^i|$ for $i < \chi$ and $\alpha < \kappa$ such that $\langle a_i^\alpha : \alpha < \kappa \rangle \models p_i$.

Then, we use strong separativity to find a witnessing sequence. That is, for each $i < \chi$, we have increasing and continuous $\langle {}^\alpha N_0^i \in K_\lambda : \alpha < \kappa \rangle$ so, for each $\alpha < \kappa$, $M \prec {}^\alpha N_0^i \prec N_0^i$ and $a_i^\alpha \in {}^{\alpha+1}N_0^i - {}^\alpha N_0^i$.

As before, we will construct $\langle M_\eta \in K_\lambda : \eta \in {}^{<\kappa}\mu \rangle$, $\langle p_j^\eta \in gS_{strsep}^1(M_\eta) : j < |gS_{strsep}^1(M_\eta)| \rangle$, $\langle i_\eta \in \chi : \eta \in {}^{<\kappa}\mu \rangle$, and $\langle \eta_i \in {}^\kappa\mu : i < \chi \rangle$ as in $(1)-(3)$ of the proof of Theorem 3.3.4 and

(4*) For $i < \chi$, a coherent, continuous $\{ {}^\alpha N_\alpha^i, \widehat{f}_{\beta,\alpha}^i : {}^\beta N_\beta^i \to_{M_{\eta_i \restriction \beta}} {}^\alpha N_\alpha^i \mid \beta < \alpha < \kappa \}$, models $\langle {}^{\alpha+1}N_\alpha^i : \alpha < \kappa \rangle$, and, for each $\beta < \alpha < \kappa$, functions

- $h_\alpha^i : {}^\alpha N_0^i \to {}^\alpha N_\alpha^i$;

- $g_{\beta+1}^i : {}^{\beta+1}N_0^i \to {}^{\beta+1}N_\beta^i$; and

- $f_{\beta+1}^i : {}^{\beta+1}N_\beta^i \to_{M_{\eta_i \restriction \beta}} {}^\beta N_\beta^i$.

These will satisfy (A), (B), and (C) from Theorem 3.3.4 and

(D) If $\alpha = \beta + 1$, then $h^i_\alpha = f^i_\alpha \circ g^i_\alpha$ and, if α is limit, then $^\alpha N^i_\alpha$ is the direct limit and, for each $\delta < \alpha$, the following commutes

$$
\begin{array}{ccc}
{}^\delta N^i_\delta & \xrightarrow{\widehat{f^i_{\delta,\alpha}}} & {}^\alpha N^i_\alpha \\
{\scriptstyle h^i_\delta}\uparrow & & \uparrow{\scriptstyle h^i_\alpha} \\
{}^\delta N^i_0 & \longrightarrow & {}^\alpha N^i_0
\end{array}
$$

(E) If $\alpha < \kappa$, then

- $h^i_\alpha \restriction {}^\alpha N^i_0 = g^i_{\alpha+1} \restriction {}^\alpha N^i_0$

- $g^i_{\alpha+1}(a^\alpha_i) \notin {}^\alpha N^i_\alpha$

- $h^i_{\alpha+1}(a^\alpha_i) = g^{i_{\eta_i \restriction \alpha}}_{\alpha+1}(a^\alpha_{i_{\eta_i \restriction \alpha}})$

- $\widehat{f^i_{\alpha,\alpha+1}} = f^i_{\alpha+1}$

Construction:

The base case and limit case are the same as in 3.3.4. In the limit, we additionally set $h^i_\alpha = \bigcup_{\beta<\alpha} h^i_\beta$.

For $\ell(\eta) = \alpha = \beta + 1$ we will apply our previous construction to the separating models. Fix some $\nu \in {}^\beta\mu$. For each $i < \chi$ such that $\eta_i \restriction \beta = \nu$, we have $h^i_\beta : {}^\beta N^i_0 \to {}^\beta N^i_\beta$. We know that $gtp(a^\beta_i/{}^\beta N^i_0, {}^{\beta+1}N^i_0)$ is big by Propositions 3.4.6 and 3.4.4. Thus we can find a big extension with domain $(h^i_\beta)^{-1}({}^\beta N^i_\beta)$. Then, applying h^i_β to this type, we get some $g^i_{\beta+1} : {}^{\beta+1}N^i_0 \to {}^{\beta+1}N^i_\beta$ so

$$
\begin{array}{ccc}
{}^\beta N^i_\beta & \longrightarrow & {}^{\beta+1}N^i_\beta \\
{\scriptstyle h^i_\beta}\uparrow & & \uparrow{\scriptstyle g^i_{\beta+1}} \\
{}^\beta N^i_0 & \longrightarrow & {}^{\beta+1}N^i_0
\end{array}
$$

commutes and $gtp(g^i_{\beta+1}(a^\beta_i)/M_\nu, {}^{\beta+1}N^i_\beta)$ is big and, therefore, strongly separative. Note that this extension uses that these types are strongly separative and not just separative. Then we can extend η_i by $\eta_i(\beta) = k$ where $k < \mu$ is the unique index such that $gtp(g^i_{\beta+1}(a^\beta_i)/M_\nu, {}^{\beta+1}N^i_\beta) = p^\nu_k$.
Then set $i_{\nu^\frown\langle i\rangle} = \min\{i < \chi : \eta_i \restriction \alpha = \nu^\frown\langle i\rangle\}$. This means that, for all $i < \chi$, we have

$$
gtp(g^i_{\beta+1}(a^\beta_i)/M_\nu, {}^{\beta+1}N^i_\beta) = gtp(g^{i_{\eta_i \restriction \alpha}}_{\beta+1}(a^\beta_{i_{\eta_i \restriction \alpha}})/M_\nu, {}^{\beta+1}N^{i_{\eta_i \restriction \alpha}}_\beta)
$$

Thus, we can find $^{\beta+1}N^i_{\beta+1} \succ {}^{\beta+1}N^{i_{\eta_i \restriction \alpha}}_\beta$ from K_λ and $f^i_{\beta+1} : {}^{\beta+1}N^i_\beta \to_{M_\nu} {}^{\beta+1}N^i_{\beta+1}$ such that $f^i_{\beta+1}(g^i_{\beta+1}(a^\beta_i)) = g^{i_{\eta_i \restriction \alpha}}_{\beta+1}(a^\beta_{i_{\eta_i \restriction \alpha}})$. Finally, set $M_{i_{\eta_i \restriction \alpha}} = {}^{\beta+1}N^{i_{\eta_i \restriction \alpha}}_\beta$ and $h^i_{\beta+1} = f^i_{\beta+1} \circ g^i_{\beta+1}$.

This is enough: For each $i < \chi$ and every $\alpha < \beta < \kappa$, we have

$$
\begin{array}{ccccccc}
M_{\eta \restriction 0} & \longrightarrow & {}^0 N^i_0 & \longrightarrow & {}^\beta N^i_0 & \longrightarrow {}^\alpha N^i_0 & \longrightarrow N^i_0 \\
\downarrow & & {\scriptstyle \widehat{f^i_{0,\beta}}}\downarrow & \swarrow {\scriptstyle h^i_\beta} & & & \\
M_{\eta \restriction \beta} & \longrightarrow & {}^\beta N^i_\beta & & & \\
\downarrow & & {\scriptstyle \widehat{f^i_{\beta,\alpha}}}\downarrow & \swarrow & {\scriptstyle h^i_\alpha} & & \\
M_{\eta \restriction \alpha} & \longrightarrow & {}^\alpha N^i_\alpha & & & &
\end{array}
$$

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that commutes. Note that this is almost the same diagram as before, except we have added the separating sequences. Then we can proceed as before, setting

1. $\widehat{M} = \cup_{\alpha < \kappa} M_{\eta \restriction \alpha}$;

2. $(\widehat{N}^i, \widehat{f}_{\alpha, \infty}) = \varinjlim_{\beta < \gamma < \kappa} ({}^\beta N_\beta^i, \widehat{f}_{\alpha, \beta}^i)$;

3. $N_1^i = \cup_{\alpha < \kappa} {}^\alpha N_0^i \prec N_0^i$.

4. $f_i : N_1^i \to \widehat{N}^i$ by $f_i = \cup_{\alpha < \kappa} (\widehat{f}_{\alpha, \infty} \circ h_\alpha^i)$; and

5. $\eta_i \in {}^\kappa \mu$ such that $i \in I_{\eta \restriction \alpha}$ for all $\alpha < \kappa$.

Then, if $\chi > \mu^\kappa$, there are $i \neq j$ such that $\eta_i = \eta_j$. As before, this would imply $p_i = p_j$, but they are all distinct. So $\chi \leq \mu^\kappa$ as desired. †

In the previous theorem, we allowed the case $\kappa = \lambda^+$. Most of the time, this is only the set-theoretic bound $strsep\mathfrak{tb}_\lambda^{\lambda^+} \leq 2^{\lambda^+}$. However, if we had $strsep\mathfrak{tb}_\lambda^1 = 1$, then we get the surprising result that $strsep\mathfrak{tb}_\lambda^{\lambda^+} \leq 1$. This will be explored along with further investigation of classifying AECs based on separative types in future work.

3.5 Saturation

We now turn from the number of infinite types to their realizations. The saturation version of Theorem 3.3.1 is much simpler to prove.

Proposition 3.5.1. *If $M \in K_\lambda$ is Galois saturated for 1-types, then M is Galois saturated for λ-types.*

Proof: Let $M_0 \prec M$ of size $< \lambda$ and $p \in gS^\lambda(M_0)$. By the definition of Galois types, there is some $N \succ M_0$ of size λ that realizes p. Find a resolution of N $\langle N_i \in K_{<\lambda} \mid i < \mathrm{cf}\,\lambda \rangle$ with $N_0 = M_0$. Then use Lemma 2.1.16 to get increasing, continuous $f_i : N_i \to M$ that fix M_0. Then $f := \cup_{i < \lambda} f_i : N \to_{M_0} M$. This implies $f(N) \models f(p) = p$ and since $f(N) \prec M$, $M \models p$. †

We can get a parameterized version with the same proof.

Proposition 3.5.2. *If $M \in K_\lambda$ is μ-Galois saturated for 1-types, then M is μ-Galois saturated for μ-types.*

The seeming simplicity of the proof of Proposition 3.5.1, especially compared to earlier uses of direct limits, hides the difficulty and complexity of the proof of Lemma 2.1.16. Although the statement is a generalization of a first order fact, its announcement was a surprise and many flawed proofs were proposed before a successful proof was given. Building on work of Shelah, Grossberg, and Kolesnikov, Baldwin [Bal09].16.5 proves a version of Lemma 2.1.16 which does not require amalgamation. This gives rise to a version of Proposition 3.5.1 in AECs even without amalgamation.

There is also a strong relationship between the value of $\mathfrak{tb}_\lambda^1$ and the existence of λ^+-saturated extensions of models of size λ. The following generalizes first order theorems like [Sh:c] Theorem VIII.4.7.

In the following theorems, we make use of a monster model, as in first order model theory, to reduce the complexity of constructions. The first relationship is clear from counting types.

Theorem 3.5.3. *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If every $M \in K_\kappa$ has an extension $N \in K_\lambda$ that is κ^+-saturated, then $\mathfrak{tb}^1_\kappa \leq \lambda$.*

Proof: Assume that every model in K_κ has a κ^+-saturated extension of size λ. Let $M \in K_\kappa$ and $N \in K_\lambda$ be that extension. Since every type over M is realized in N, we have $|gS(M)| \leq \|N\| = \lambda$. Taking the sup over all $M \in K_\kappa$, we get $\mathfrak{tb}^1_\kappa \leq \lambda$, as desired. †

Going the other way, we have both a set theoretic hypothesis and model theoretic hypothesis that imply instances of a κ^+-saturated extension. The set-theoretic version is well known.

Theorem 3.5.4. *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If $\lambda^\kappa = \lambda$, then every $M \in K_\kappa$ has an extension $N \in K_\lambda$ that is κ^+ saturated.*

Note that the hypothesis implies $\mathfrak{tb}^1_\lambda \leq \lambda$. Without this set theoretic hypothesis, reaching our desired conclusion is much harder. $\lambda^\kappa = \lambda$ means that we can consider all κ size submodels of a λ sized model without going up in size. Without this assumption, things become much more difficult and we must rely on model theoretic hypotheses. The following has a stability-like hypothesis, sometimes called 'weak stability;' see [JrSh875], for instance.

Theorem 3.5.5. *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If $\mathfrak{tb}^1_\kappa \leq \kappa^+$, then every $M \in K_\kappa$ has an extension $N \in K_{\kappa^+}$ that is saturated.*

Proof: We proceed by a series of increasingly strong constructions.
Construction 1: For all $M \in K_\kappa$, there is $M^* \in K_{\kappa^+}$ such that all of $S^1(M)$ is realized in M^*.
This is easy with $|S^1(M)| \leq \kappa^+$.
Construction 2: For all $M \prec N$ from K_κ and $M \prec M' \in K_{\kappa^+}$, there is some $N' = *(M, N, M') \in K_{\kappa^+}$ such that $N, M' \prec N'$ and all of $S^1(N)$ are realized in N'.
For each $p \in S^1(N)$, find some $a_p \in |\mathfrak{C}|$ that realizes it. Then find some $N' \prec \mathfrak{C}$ that contains $\{a_p : p \in S^1(N)\} \cup |M'| \cup |N|$ of size κ^+. This is possible since $|S^1(N)| \leq \kappa^+$.
Construction 3: For all $M \in K_{\kappa^+}$ there is some $M^+ \in K_{\kappa^+}$ such that $M \prec M^+$ and, if $M_0 \prec M$ of size κ, then all of $S^1(M_0)$ are realized in M^+.
Find a resolution $\langle M_i : i < \kappa^+ \rangle$ of M. Set $N_0 = (M_0)^*$, $N_{i+1} = *(M_i, M_{i+1}, N_i)$, and take unions at limits. Then $M^+ = \cup_{i < \kappa^+} N_i$ works.
Construction 4: For all $M \in K_{\kappa^+}$, there is some $M^\# \in K_{\kappa^+}$ such that $M \prec M^\#$ and $M^\#$ is saturated.
Let $M \in K_\kappa$. Set $M_0 = M$, $M_{i+1} = (M_i)^+$, and take unions at limits. Then $M^\# = M_{\kappa^+}$ is saturated. Then, to prove the proposition, let $M \in K_\kappa$. Since K has no maximal model, it has an extension M' in K_{κ^+}. Then $(M')^\#$ is the desired saturated extension of M. †

3.6 Further Work

In working with Galois types, as we do here, the assumption of amalgamation simplifies the definitions and construction by making E_{AT} already an equivalence relation in the definition of types; see Definition 2.1.12. Could we obtain some bound on the number of long types in the absence of amalgamation?

In this paper, we introduced the definitions of S_{sep} and S_{strsep}. There are several basic questions to explore.

First, is S_{strsep} necessary? That is, is there an AEC where the size of S_{strsep} is well behaved, but S_{sep} is chaotic as in Proposition 3.4.1; or can the proof of Theorem 3.4.8 be improved to provide the same bound on $seq\mathfrak{tb}$? By Proposition 3.4.5, any example of this chaotic behavior must have amalgamation but not disjoint amalgamation. Perhaps one of the examples from [BKS927] can be refined for this purpose.

In the original definitions of separative and strongly separative types, any ordered set I was allowed as an index set and the separation properties were required to hold for all subsets instead of just an initial segment. We give the original definition here under the names *unordered separative* and *unordered strongly separative types*.

Definition 3.6.1.

1. *For $M \in K$, define $gS_{usep}^{\alpha}(M) = \{gtp(\langle a_i : i \in I \rangle / M, N) \in gS_{na}^{I}(M) :$ for all $I_0 \subset I$, there is some $M \prec N_{I_0} \prec N$ such that $a_i \in N_{I_0}$ iff $i \in I_0\}$.*

2. *For $M \in K$, define $gS_{ustrsep}^{I}(M) = \{p = gtp(\langle a_i : i \in I \rangle / M, N) \in gS_{usep}^{I}(M) :$ for all $I_0 \subset I$ and $M \prec N_{I_0} \prec N$, if with $a_i \in N_{I_0}$ iff $i \in I_0$, then for every $f : N_{I_0} \to N_1^+$ with $\|N_1^+\| = N_{I_0}$, there is some $g : N \to N_2^+$ such that $f \subset g$ and $g(\langle a_i : i \in I - I_0 \rangle) \notin N_1^+\}$.*

Are these definitions equivalent to the ones given in Section 3.4, or is there some example of an AEC where the two notions are distinct? This question will likely be clarified by a lower bound for $strsep\mathfrak{tb}$ or an example lacking amalgamation.

A more lofty goal would be to attempt to classify stable, DAP AECs by the possible values of $nat\mathfrak{tb}_{\lambda}^{1} = sep\mathfrak{tb}_{\lambda}^{1}$. That is, we know that $nat\mathfrak{tb}_{\lambda}^{1}$ is a cardinal between 1 and λ and that this controls the value of $sep\mathfrak{tb}_{\lambda}^{\kappa}$ for all $\kappa \leq \lambda$. For each value in $[1, \lambda] \cap CARD$, does an AEC with DAP and exactly that many non-algebraic types of length one exist? Of particular interest is the discussion after Theorem 3.4.8. The conclusion of only one separative type (or strongly separative type, if we wish to drop the assumption of DAP) of any length over a model seems to be a very powerful hypothesis.

Looking back to the first order case, it would be interesting to find a syntactic characterization of separative types, keeping in mind that separative and strongly separative types are equivalent in this context.

Chapter 4

Tameness and Large Cardinals

4.1 Introduction

In this chapter, we investigate the relationship between large cardinals and tameness. Our main result (Theorem 4.2.5) is the following

Theorem 4.1.1. *If K is an AEC with $LS(K) < \kappa$ and κ is strongly compact, then K is κ-tame.*

This is combined with the result of Grossberg and VanDieren to give us the consistency of Shelah's Eventual Categoricity Conjecture for Successors. Section 4.5 provides more details, including the derivation of amalgamation. This result also improves [MaSh285] by showing that their categoricity result holds even for AECs that are not axiomatized by an infinitary theory.

The above result is part of a larger investigation of tame AECs that began with Grossberg and VanDieren's introduction of tameness in [GV06b]. In the introduction of [GV06a], Grossberg and VanDieren list several previously studied nonelementary classes that turn out to be tame. This list includes previous AECs for which a classification theory exists. This lead them to the following conjecture about categoricity and tameness.

Conjecture 4.1.2. *[Grossberg-VanDieren] Suppose K is an AEC. If K is categorical in some $\lambda \geq Hanf(LS(K))$ (or some other value depending only on $LS(K)$), then there exists $\chi < Hanf(LS(K))$ such that K is χ-tame.*

Our main theorem can be seen as proving a stronger version of this from the existence of a strongly compact cardinal instead of the categoricity assumption.

Some assumption (categoricity, large cardinals, etc.) is known to be necessary for any theorem that concludes tameness for all AECs. This follows from the examples of nontame AECs mentioned in the introduction: the Hart and Shelah [HaSh323] examples and the Baldwin and Shelah examples [BlSh862]. Section 4.6 further discussion these examples.

In addition to tameness, we use a dual locality property called *type shortness*. This property is defined explicitly in Chapter II, but briefly says that if two types of long, infinite sequences indexed by the same set differ, then there is a short subsequence where they already differ. Comparing this with tameness, we are replacing the condition on the domain of the type with a condition on the index of the type. The combination of these properties can be used to obtain a new notion of nonforking that can be seen as an AEC analogue of coheir; this is done in the next chapter.

We now outline the chapter. The main results of this chapter are in Sections 4.2, 4.3, and 4.4. Each of these sections assumes a different large cardinal axiom and uses a different technique to prove various levels of type shortness and tameness: Section 4.2 uses the ultrafilter definition of a strongly compact cardinal; Section 4.3 uses the elementary embedding definition of a measurable cardinall and Section 4.4 uses the indescribability definition of a weakly compact cardinal. Section 4.5 combines the results from this paper with the papers mentioned in the introduction. This section contains Theorem 4.5.5, the consistency of Shelah's Eventual Categoricity Conjecture for Successors. Finally, Section 4.6 poses some new questions, especially in the area of the large cardinal strength of different universal tameness properties.

Before proceeding, we offer another definition. Given a large cardinal κ, we want to identify the class of AECs which are closed under the elementary embedding or other structure coming from their large cardinal definition. Typically, this occurs when the AEC is described by a definition "smaller" than κ.

Definition 4.1.3. *For a cardinal κ, we say that an AEC is* essentially below κ *iff a) $LS(K) < \kappa$ or b) $K = (Mod\, T, \prec_{\mathcal{F}})$ for T a $L_{\kappa,\omega}$ theory.*

In either case, the key pieces of the definition exist below κ; in the case of a theory in $L_{\kappa,\omega}$, the theory itself might consist of more than κ-many sentences, but each sentence is defined from κ-many pieces.

There are other properties of AECs that assert different locality properties of types. [BlSh862] contains some of these. The arguments in the following sections are also useful in deriving those properties.

After seeing a preliminary version of this work, Jose Iovino pointed us to the work on Metric Abstract Elementary Classes (MAECs) by Hirvonen and Hyttinen [HH09] and others. This is a more general framework that extends AECs as continuous first-order logic extends first-order logic and is more suited for dealing with analytic concepts like being a complete metric space. Although there is a slightly different notion of ultraproducts, the theorems of this chapter still hold in that context.

4.2 Strongly Compact

We begin with a study of AECs under the assumptions that there is a strongly compact cardinal κ and a given AEC is essentially below κ (see Definition 4.1.3), but has a model above κ. Since κ is strongly inaccessible, this is equivalent to the AEC having a model above its Hanf number.

Definition 4.2.1 ([Jec06].20). *An uncountable cardinal κ is* strongly compact *iff every κ-complete filter can be extended to a κ-complete ultrafilter.*
Equivalently, $L_{\kappa,\omega}$ and $L_{\kappa,\kappa}$ satisfy the compactness theorem.
Equivalently, for every $\lambda \geq \kappa$, there is some elementary (in the first-order sense) embedding $j : V \to \mathcal{M}$ with critical point κ such that $j(\kappa) > \lambda$ and there is some $Y \in \mathcal{M}$ of size λ such that $j''\lambda \subset Y$.
Equivalently, for every $\lambda \geq \kappa$, there is a fine, κ complete ultrafilter U on $P_\kappa \lambda$; that is, a κ complete ultrafilter so, for every $\alpha < \kappa$, we have $[\alpha] = \{X \in P_\kappa \lambda : \alpha \in X\} \in U$.

In this section, we prefer to use the latter ultrafilter formulation because it is more model-theoretic in nature. In the next section, on measurable cardinals, we discuss the elementary embedding formulation of a large cardinal that is preferred by set theorists.

The most basic and fundamental model-theoretic fact about ultraproducts is Łoś' Theorem, which tells us that $Mod\, T$ is closed under ultraproducts. We wish to prove a version of Łoś' Theorem for AECs. This will generalize the version for $L_{\kappa,\omega}$ when κ is measureable.

Theorem 4.2.2 (Łoś' Theorem for $L_{\kappa,\omega}$). *Let U be a κ-complete ultrafilter over I, L be a language, and $\langle M_i : i \in I \rangle$ be L structures. Then, for any $[f_1]_U, \dots [f_n]_U \in \Pi M_i / U$ and $\phi(x_1, \dots, x_n) \in L_{\kappa,\omega}$, we have*

$$\Pi M_i / U \models \phi([f_1]_U, \dots, [f_n]_U) \text{ iff } \{i \in I : M_i \models \phi(f_1(i), \dots, f_n(i))\} \in U$$

This is proved similarly to the first-order version. Our version for AECs is necessarily more complex since we do not have any syntax. Thus, the characterization must be done semantically. However, the following theorem aims to obtain the same results as the first-order version. Of particular interest are parts (5) and (6): (5) says that if $M = \cup_{i<\kappa} M_i$ for $\langle M_i : i < \kappa \rangle$ increasing, then we can cannonically embed M into $\Pi M_i / U$ and (6) says the same thing for $\langle M_i : i < \kappa \rangle$ a directed set.

Theorem 4.2.3 (Łoś' Theorem for AECs). *Suppose K is an AEC essentially below κ and U is a κ-complete ultrafilter on I. Then K and the class of K-embeddings is closed under κ-complete ultrapowers and the ultrapower embedding. In particular,*

1. *if $\langle M_i \in K : i \in I \rangle$, then $\Pi M_i / U \in K$;*

2. *if $\langle M_i \in K : i \in I \rangle, \langle N_i \in K : i \in I \rangle$ and, for every $i \in I$, $M_i \prec_K N_i$, then $\Pi M_i / U \prec_K \Pi N_i / U$*

3. *if $\langle M_i \in K : i \in \rangle, \langle N_i \in K : i \in I \rangle$ and, for every $i \in I$, there is some $h_i : M_i \cong N_i$, then $\Pi h_i : \Pi M_i / U \cong \Pi N_i / U$, where Πh_i is defined by taking $[i \mapsto f(i)]_U \in \Pi M_i / U$ to $[i \mapsto h_i(f(i))]_U \in \Pi N_i / U$;*

4. *if $\langle M_i \in K : i \in I \rangle, \langle N_i \in K : i \in I \rangle$ and, for every $i \in I$, there is some $h_i : M_i \to N_i$, then $\Pi h_i : \Pi M_i / U \to \Pi N_i / U$, where Πh_i is defined by taking $[i \mapsto f(i)]_U \in \Pi M_i / U$ to $[i \mapsto h_i(f(i))]_U \in \Pi N_i / U$;*

5. *if $I = \kappa$ and $\langle M_i \in K : i < \kappa \rangle$ is an increasing sequence, then the ultrapower embedding $h : \bigcup_{i < \kappa} M_i \to \Pi M_i / U$ defined as $h(m) = [f_m]_U$, where*

$$f_m(i) = \begin{cases} m & \text{if } m \in |M_i| \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

is a K-embedding; and

6. *if $\langle M_i \in K : i \in I \rangle$ is a directed set, so in particular $M := \bigcup_{i \in I} M_i \in K$ and, for all $m \in |M|$, we have $[m] = \{i \in I : m \in M_i\} \in U$, then the ultrapower embedding $h : M \to \Pi M_i / U$ is a K-embedding, where $h(m) = [f_m]_U$ and*

$$f_m(i) = \begin{cases} m & \text{if } m \in |M_i| \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

Proof: If K is an AEC essentially below κ, then either it is a model of an $L_{\kappa,\omega}$ theory or $LS(K) < \kappa$. In the first case, this follows from Łoś' Theorem for $L_{\kappa,\omega}$.

If $LS(K) < \kappa$, then Shelah's Presentation Theorem above says that $K = PC(T_1, \Gamma, L(K))$ for $|T_1| = LS(K) < \kappa$. During the following proofs, we use the fact observed at [Sh:c].VI.0.2 that an ultraproduct of reducts is the reduct of the ultraproducts.

1. Each $M_i \in K = PC(T_1, \Gamma, L(K))$, there is some $L(T_1)$ structure $M_i^* \in EC(T_1, \Gamma)$ such that $M_i = M_i^* \upharpoonright L(K)$. Then $\Pi M_i / U = \Pi(M_i^* \upharpoonright L(K))/U = \Pi M_i^* / U \upharpoonright L(K)$, such that $\Pi M_i / U$ is the restriction to $L(K)$ of a $L(T_1)$ structure. Furthermore, there is an $L_{\kappa,\omega}$ sentence ψ st, for any $L(T_1)$ structure M, $M \models \psi$ iff $M \in EC(T_1, \Gamma)$. Thus, for all $i \in I$, $M_i^* \models \psi$. So by Łoś' Theorem for $L_{\kappa,\omega}$, $\Pi M_i^* / U \models \psi$. Thus, $\Pi M_i^* / U \in EC(T_1, \Gamma)$ and $\Pi M_i / U = \Pi M_i^* / U \upharpoonright L(K) \in PC(T_1, \Gamma, L(K)) = K$.

2. From Shelah's Presentation Theorem, for each N_i, there are , $M_i^*, N_i^* \in EC(T_1, \Gamma)$ such that $M_i^* \upharpoonright L(K) = M_i$, $N_i^* \upharpoonright L(K) = N_i$, and $M_i^* \subseteq N_i^*$. By the above part, $\Pi M_i^* / U, \Pi N_i^* / U \in$

$EC(T_1, \Gamma)$ and, by the definition of an ultraproduct, $\Pi M_i^*/U \subseteq \Pi N_i^*/U$. Again applying Shelah's Presentation Theorem, we get that

$$(\Pi M_i^*/U) \upharpoonright L(K) = \Pi M_i/U \prec_K \Pi N_i/U = (\Pi N_i^*/U) \upharpoonright L(K)$$

3. First we note that Πh_i is a bijection. If $[f]_U \in \Pi N_i/U$, then $[i \mapsto h_i^{-1}(f(i))]_U \in \Pi M_i/U$ and $\Pi h_i([i \to h_i^{-1}(f(i))]_U) = [f]_U$. If $[f]_U \neq [g]_U \in \Pi M_i/U$, then $\{i \in I : f(i) = g(i)\} \notin U$. But this left hand side is $\{i \in I : h_i(f(i)) = h_i(g(i))\}$, so $\Pi h_i([f]_U) \neq \Pi h_i([g]_U)$.

   Now we must show that it respects $L(K)$. Suppose that $R \in L(K)$ is an n-ary relation. Then

$$\Pi M_i/U \models R([f_1]_U, \ldots, [f_n]_U)$$
$$\{i \in I : M_i \models R(f_1(i), \ldots, f_n(i))\} \in U$$
$$\{i \in I : N_i \models R(h_i(f_1(i)), \ldots, h_i(f_n(i)))\} \in U$$
$$\Pi N_i/U \models R(\Pi h_i/U([f_1]_U), \ldots, \Pi h_i/U([f_n]_U))$$

   as desired. The same proof works for functions, or assume $L(K)$ is relational by replacing functions with their graph.

4. For each $i \in I$, we have a $h_i : M_i \cong h_i(M_i)$ with $h_i(M_i) \prec_K N_i$; see the definition of a K-embedding. From above, we know that $\Pi h_i(M_i)/U \prec \Pi N_i/U$ and $\Pi h_i : \Pi M_i/U \cong \Pi h_i(M_i)/U$. So by the definition of a K embedding, we have our conclusion.

5. This follows from the next one. Note that, by κ completeness, $[m] = \{\alpha < \kappa : \alpha \geq \beta\} \in U$, where $\beta = \min\{\gamma < \kappa : m \in |M_\gamma|\}$.

6. Since we modulus by U, the definition of f_m only matters on a measure one set, namely $[m]$. We proceed as in (1) and (2). We can extend each M_i to $M_i^* \in EC(T_1, \Gamma)$. Then $\Pi M_i/U = \Pi(M_i^* \upharpoonright L)/U = (\Pi M_i^*/U) \upharpoonright L$. Then this induces an $L(T_1)$ expansion of $h(M)$ called $h(M)^* \subseteq \Pi M_i^* \upharpoonright L/U$. So $h : M \to \Pi M_i/U$. $\qquad\qquad\dagger$

Note, in particular, that in (3) and (4), we have defined the 'ultraproduct' of a series of embeddings. We will generally refer to this as the average of those embeddings and will later use this fact in particular when $N \in K$ and we have many $f_i \in Aut N$; then we know that $\Pi f_i \in Aut \Pi N/U$.

In our definition of essentially below, we hoped to capture all AECs that are closed under complete enough ultraproducts in the sense above. However, this is not the case: we could take an AEC K which is essentially below κ and form the AEC $K' = (K_{\kappa+})^{up}$ (recall Definition 2.1.7) by taking out all models of size κ or smaller. Then K' is not essentially below κ, but is still closed under κ-complete ultraproducts.

However, the hypothesis of an AEC which is essentially below κ is natural and somewhat tight, in the sense that there are simple examples of AECs that just fail to be essentially below κ and are not closed under κ-complete ultraproducts. The following example mirrors the construction of nonstandard models of PA.

Example 4.2.4. *Let $L = \{<, c_\alpha\}_{\alpha < \kappa}$ and set $\psi \in L_{\kappa^+, \omega}$ to be the sentence*

$$\text{``$<$ is a linear order''} \wedge \forall x (\vee_{\alpha < \kappa} x = c_\alpha) \wedge (\bigwedge_{\alpha < \beta < \kappa} c_\alpha < c_\beta)$$

Let $\mathcal{F}$ be a κ-sized fragment containing ψ. Then $K = (Mod\,\psi, \prec_{\mathcal{F}})$ is an AEC with $LS(K) = \kappa$, so it 'just fails' to be essentially below κ. Any $M \in K$ is isomorphic to $(\kappa, \in, \alpha)_{\alpha<\kappa}$. Thus, K is not closed under κ-complete ultraproducts.

The results in this chapter that have a hypothesis of "essentially below" will all continue to hold in any AEC that is closed under sufficiently closed ultraproducts.

Now we are ready to establish the main theorem of this section, that AECs that are essentially below a strongly compact cardinal are tame and type short. This allows us to connect our large cardinal assumptions to known model theoretic properties. Afterwards, we will continue our investigation of ultraproducts of AECs; these results will make more sense in light of the fact that types are determined by their $< \kappa$ restrictions.

In this theorem, we assume that K has a monster model. However, this is not necessary and we do not even need to assume amalgamation for the conclusion. We include the stronger assumptions to simplify the proof, but provide Theorem 4.3.4 as a "proof of concept" that this assumption can be removed.

Theorem 4.2.5. *Suppose K is essentially below κ, κ is strongly compact, and K has amalgamation, joint embedding, and no maximal models. Then types are determined by the restrictions of their domain to $< (\kappa + LS(K)^+)$-sized models and their length to $< \kappa$-sized sets; that is, given $M \in K$ and $p, q \in S^I(M)$,*

$$\text{if } p^{I_0} \upharpoonright M_0 = q^{I_0} \upharpoonright M_0 \text{ for all } I_0 \in P_\kappa I \text{ and } M_0 \in P^*_{\kappa + LS(K)^+} M, \text{ then } p = q.$$

The inclusion of '$+LS(K)^+$' is needed for the case that K is the class of models of some theory in a fragment $\mathcal{F}$ of $L_{\kappa,\omega}$ with $LS(K) = |\mathcal{F}| \geq \kappa$; in this case, it would be impossible for K to be $< \kappa$ tame because there would be no models of size $< \kappa$.

First, we prove a technical lemma. In our proof of Theorem 4.2.5, there is a place where we will want to take an ultraproduct of our monster model. However, this would run counter to our intuition of the monster model containing all models since the monster model cannot contain its own ultraproduct. To avoid this, we introduce a smaller model that functions as the monster model exactly as we need, but without any blanket assumptions of containing all models or being model homogeneous. We call such a model a *local monster model*.

Lemma 4.2.6 (Local Monster Model). *Suppose we have some collection $\{M_i \in K_{\leq\mu} : i < \mu\}$ and $\{f_i \in Aut\mathfrak{C} : i < \mu\}$ such that each $M_i \prec \mathfrak{C}$. Then there is some $\mathcal{N} \in K_\mu$ such that for each $i < \mu$ we have $M_i \prec \mathcal{N}$ and $f_i \upharpoonright \mathcal{N} \in Aut\mathcal{N}$.*

Proof: Let $N_0 \prec \mathfrak{C}$ of size μ such that $\bigcup_{i<\mu} |M_i| \subset |N_0|$. Then each $M_i \prec N_0$. For $n < \omega$, if we have N_n, set $N_{n+1} \prec \mathfrak{C}$ to be of size μ such that it contains $|N_n| \cup \bigcup_{i<\mu}(f_i[N_n] \cup f_i^{-1}[N_n])$. Then set $\mathcal{N} = \cup_{n<\omega} N_n$. †

Proof of Theorem 4.2.5: Let $p, q \in S^I(M)$ be as above. Find $X = \langle x_i : i \in I \rangle \models p$ and $Y = \langle y_i : i \in I \rangle \models q$. Then, by Lemma 4.2.6, we find a local monster model $\mathcal{N}$ such that, for all $(I_0, M_0) \in P_\kappa I \times P^*_{\kappa+LS(K)^+} M$, there is some $f_{(I_0, M_0)} \in Aut_{M_0}\mathcal{N}$ such that $f_{(I_0, M_0)}(x_i) = y_i$ for all $i \in I_0$. Next, by the final equivalent definitions of strongly compact cardinals from Definition 4.2.1, we find a fine, κ complete ultrafilter U on $P_\kappa I \times P^*_{\kappa+LS(K)^+} M$; that is, one such that $[(i, m)] = \{(I_0, M_0) \in$

$P_\kappa I \times P^*_{\kappa+LS(K)^+} M : i \in I_0, m \in M_0\} \in U$ for all $i \in I$ and $m \in M$.

Then, by Theorem 4.2.3.6, $\Pi\mathcal{N}/U \in K$ and our average of these automorphisms $f \in Aut\Pi\mathcal{N}/U$. Recall that f takes $[(I_0, M_0) \to g(I_0, M_0)]_U$ to $[(I_0, M_0) \to f_{(I_0, M_0)}(g(I_0, M_0))]_U$. Now we must prove two claims

- f **fixes** $h(M)$

  Let $m \in |M|$. Given any $i \in I$, $[(i, m)] \in U$ and, if $(I_0, M_0) \in [(i, m)]$, then $m \in M_0$, such $f_{(I_0, M_0)}(m) = m$. Thus,

  $$[(i, m)] \subset \{(I_0, M_0) \in P_\kappa I \times P^*_{\kappa+LS(K)^+} M : f_{(I_0, M_0)}(m) = m\} \in U$$

  and $f \circ h(m) = [(I_0, M_0) \to f_{(I_0, M_0)}(m)]_U = [(I_0, M_0) \to m]_U = h(m)$.

- $f(h(x_i)) = h(y_i)$ **for every** $i \in I$

  Let $i \in I$. Given any $m \in |M|$, $[(i, m)] \in U$ and, if $(I_0, M_0) \in [(i, m)]$, then $i \in I_0$, so $f_{(I_0, M_0)}(x_i) = y_i$. Thus,

  $$[(i, m)] \subset \{(I_0, M_0) \in P_\kappa I \times P^*_{\kappa+LS(K)^+} M : f_{(I_0, M_0)}(x_i) = y_i\} \in U$$

  and $f \circ h(x_i) = [(I_0, M_0) \to f_{(I_0, M_0)}(x_i)]_U = [(I_0, M_0) \to y_i]_U = h(y_i)$.

Now we have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\ f \circ h\ } & \Pi\mathcal{N}/U \\
\uparrow & & \uparrow{\scriptstyle h} \\
M & \longrightarrow & \mathcal{N}
\end{array}
$$

with $f \circ h(x_i) = h(y_i)$ for all $i \in I$. Thus, $p = q$. $\qquad\qquad$ †

The above theorem can be interpreted as saying that if we have two different types, then they are different on a "formula," if we take formula to mean a type of $< \kappa$ length over a domain of size $< \kappa$. With this definition of formula, we can replace a large type by the set consisting of all of its small restrictions and type equality will be preserved. In the rest of this section, we will see that, since κ is strongly compact, this notion of formulas as small types will be fruitful. We now return to the development of our ultraproducts with a version of Łoś' Theorem.

Also, we strengthen our hypothesis to $LS(K) < \kappa$ instead of just K essentially below κ. This is because [MaSh285].2.10 has shown that, with a monster model, Galois types in models of a $L_{\kappa,\omega}$ theory correspond to consistent sets of formulas from a fragment of $L_{\kappa,\kappa}$, so the following results are already known.

Note that the following theorem only requires a measurable cardinal.

Theorem 4.2.7 (Łoś' Theorem for AECs, part 2). *Suppose that K is an AEC with amalgamation, joint embedding, and no maximal models and κ is a measurable cardinal such that $\kappa < LS(K)$. Let $N^- \prec N \in K$ and $p \in S(N^-)$ with $\|N^-\| + \ell(p) < \kappa$ and U be a κ-complete ultrafilter on I. Then $[g]_U \in \Pi N/U$ realizes $h(p)$ iff $\{i \in I : g(i) \vDash p\} \in U$, where $h : N \to \Pi N/U$ is the canonical ultrapower embedding.*

Proof: $\Leftarrow$ Suppose that $[g]_U \in \Pi N/U$ with $X := \{i \in I : g(i) \vDash p\} \in U$. Let $a \vDash p$. By Lemma 4.2.6, there is a local monster model $\mathcal{N}$ such that, for each $i \in X$, there is $f_i \in Aut_{N^-}\mathcal{N}$ such that $f_i(g(i)) = a$.

Define $f^+ : \Pi\mathcal{N}/U \to \Pi\mathcal{N}/U$ to be the average of these maps. That is, f^+ of $[i \mapsto k(i)]_U$ is $[i \mapsto f_i(k(i))]_U$; although these f_i's don't exist everywhere, they exist on a U-large set and this is enough. Then, by Łoś' Theorem for AECs, $f^+ \in Aut_{h(N^-)}\Pi\mathcal{N}/U$. Note that $h(N^-)$ is $h''N^-$ and not $\Pi N^-/U$. Also, f_i sends $g(i)$ to a on a large set, so $f^+([g]_U) = h(a)$. So $[g]_U$ realizes $tp(h(a)/h(N^-)) = h(p)$ as desired.

$\Rightarrow$ Let $[g]_U \in \Pi N/U$ realize $h(p) \in S(h(N^-))$. For each $q \in S(N^-)$, set $X_q = \{i \in I : g(i) \text{ realizes } q\}$. Since different q's are mutually exclusive, these are all disjoint and they partition I. We easily have $|S(N^-)| \leq 2^{\|N^-\|} < \kappa$. Since U is κ-complete, this means that, for some $q_0 \in S(N^-)$, $X_{q_0} \in U$. By the previous direction, that means that $[g]_U$ realizes $h(q_0)$. But by assumption, the type of $[g]_U$ over $h(N^-)$ is $h(p)$, so $p = q_0$. Thus $X_p = \{i \in I : g(i) \text{ realizes } p\} \in U$, as desired. †

Now that we have Łoś' Theorem, we prove a companion result to Theorem 4.2.5. This motivated our conception of types as sets of smaller types or "formulas." Here we show that, as with the first-order case, any consistent set of formulas can be completed to a type, even when the set is incomplete.

We introduce some notation to make this as general as possible. Even with a strongly compact cardinal, a key difference between small types and formulas is that there is no negation of a type: given a type p, there is no type q such that all elements realize either p or q and not both. To compensate for this, we want to allow specification of both types to be realized and types to be avoided. In the following, X represents the types to be realized and $\neg X$ represents the types to be avoided.

Definition 4.2.8. *Fix $M \in K$ and I a linear order. Let $X \subset \{p \in S^{I_0}(M^-) : I_0 \in P_\kappa I \text{ and } M^- \in P_\kappa^* M\}$ and $\neg X \subset \{q \in S^{I_0}(M^-) : I_0 \in P_\kappa I \text{ and } M^- \in P_\kappa^* M\}$.*

- *We say that $\mathbf{a} = \langle a_i : i \in I \rangle$ realizes $(X, \neg X)$, written $\mathbf{a} \vDash (X, \neg X)$ iff, for every $p \in X$, $\langle a_i : i \in \ell(p) \rangle \vDash p$ and, for every $q \in \neg X$, $\langle a_i : i \in \ell(q) \rangle \nvDash q$. We say that $(X, \neg X)$ is consistent iff it has a realization.*

- *We say that $(X, \neg X)$ is $< \kappa$ consistent iff $(X_0, \neg X_0)$ is consistent for every $X_0 \in P_\kappa X$ and $\neg X_0 \in P_\kappa \neg X$.*

Theorem 4.2.9. *Suppose K is an AEC with amalgamation, joint emebedding, and no maximal models and κ is strongly compact such that $LS(K) < \kappa$. Let $M \in K$ and I be a linear order. Given $X \subset \{p \in S^{I_0}(M^-) : I_0 \in P_\kappa I \text{ and } M^- \in P_\kappa^* M\}$ and $\neg X \subset \{q \in S^{I_0}(M^-) : I_0 \in P_\kappa I \text{ and } M^- \in P_\kappa^* M\}$, $(X, \neg X)$ is consistent iff it is $< \kappa$ consistent.*

Proof: One direction is obvious, so suppose that $(X, \neg X)$ is $< \kappa$ consistent. For every $N \in P_\kappa^* M$, let $X_N = \{p \in X : \text{dom } p \prec N\}$ and $\neg X_N = \{q \in \neg X : \text{dom } q \prec N\}$. Then, by assumption $(X_N, \neg X_N)$ is consistent, so there is $\mathbf{a}_N = \langle a_i^N : i \in I \rangle$ that realizes $(X_N, \neg X_N)$; if $X_N = \neg X_N = \emptyset$, then pick $\mathbf{a}_N$ arbitrarily. Let $M^+ \succ M$ contain all $\mathbf{a}_N$ and let U be a κ complete, fine ultrafilter on $P_\kappa^* M$. Recall that $h : M^+ \to \Pi M^+/U$ is the canonical embedding.

For each $i \in I$, set $a_i := [N \mapsto a_i^N]_U$ for $i \in I$ and set $\mathbf{a} := \langle a_i : i \in I \rangle$. We claim that $\mathbf{a} \vDash h(X, \neg X) = (\{h(p) : p \in X\}, \{h(q) : q \in \neg X\})$. Suppose $p \in X$ and $M^- \prec M$ and $I_0 \subset I$ such

that $p \in S^{I_0}(M^-)$. Then

$$[M^-] = \{N \in P_\kappa^* M : M^- \prec M\} \subset \{N \in P_\kappa^* M : \langle a_i^N : i \in I_0 \rangle \vDash p\}$$

by construction. Since this first set is in U by fineness, $\langle a_i : i \in I_0 \rangle \vDash h(p)$ by the previous theorem. Now suppose $q \in \neg X$ and $M^- \prec M$ and $I_0 \subset I$ such that $q \in S^{I_0}(M^-)$. For contradiction, suppose that $\langle a_i : i \in I_0 \rangle \vDash h(q)$. Then, by the previous theorem, $\{N \in P_\kappa^* M : \langle a_i^N : i \in I_0 \rangle \vDash q\} \in U$. Then let $N' \in \{N \in P_\kappa^* M : \langle a_i^N : i \in I_0 \rangle \vDash q\} \cap [M^-]$; this intersection is nonempty because it is in U. Then $\langle a_i^{N'} : i \in I_0 \rangle$ both realizes and does not realize q, a contradiction. Thus, $\langle a_i : i \in I_0 \rangle$ does not realize $h(q)$ and we have shown $\mathbf{a} \vDash h(X, \neg X)$, as desired.

Let h^+ be an $L(K)$ isomorphism that extends h and has image $\Pi M^+ / U$. Then $(h^+)^{-1}(\mathbf{a})$ witnesses the consistency of $(X, \neg X)$. †

We can use a similar argument to transfer saturation from M to $\Pi M / U$.

Theorem 4.2.10. *Suppose K is an AEC with amalgamation, joint embedding, and no maximal models and κ is strongly compact such that $LS(K) < \kappa$. For all $M \in K$ and linear order I, there is some κ complete U such that, for any $p \in S^I(M)$ that has all $< \kappa$ restrictions realized in M, $\Pi M / U \models h(p)$.*

Proof: Let U be a κ complete, fine ultrafilter on $P_\kappa I \times P_\kappa^* M$ and, for each small approximation $p^{I_0} \upharpoonright M^-$, pick some $\mathbf{a}^{(I_0, M^-)} := \langle a_i^{(I_0, M^-)} \in M : i \in I_0 \rangle \vDash p^{I_0} \upharpoonright M^-$. Now consider the sequence $\langle [(I_0, M_0) \mapsto a_i^{(I_0, M^-)}]_U : i \in I \rangle$. This sequence is in $\Pi M / U$ since each $a_i^{(I_0, M_0)} \in M$. By the same argument as the previous theorem, this sequence realizes $h(p)$. †

Corollary 4.2.11. *If $M \in K$ is $< \kappa$ saturated, then there is some κ-complete U such $\Pi M / U$ realizes all types over M.*

4.3 Measurable

We now turn our attention to what happens if our large cardinal is only measurable.

Definition 4.3.1 ([Jec06].17)**.** *An uncountable cardinal κ is* measurable *iff there is a normal, κ-complete ultrafilter on κ. Equivalently, there is some elementary embedding $j : V \to \mathcal{M}$ with critical point κ such that $^\kappa \mathcal{M} \subset \mathcal{M}$.*

Unsurprisingly, we don't get as strong results here. Instead, we just get results of $(< \lambda, \lambda)$-tameness and type shortness whenever cf $\lambda = \kappa$. Reexamining the above proof, an argument readily presents itself by using the κ-complete ultrafilter on κ and redoing the above arguments. Instead of repeating the above proof, we prove this theorem in two different ways: once with a monster model and using the ultrapower definition, and the second time using ultrafilters but no assumption of amalgamation at all. We do these proofs in order to showcase different large cardinal techniques on AECs. The use of the elementary embedding is of particular interest, because this is the formulation of large cardinals most studied by modern set theorists and will hopefully shed light on future work in this direction, while the proof without amalgamation shows the we get the results from just large cardinals and do not need additional, structural assumptions on K, like amalgamation.

Theorem 4.3.2. *Suppose K is an AEC essentially below κ measurable with amalgamation, joint emebedding, and no maximal models. Let $M = \bigcup_{\alpha < \kappa} M_\alpha$ and $I = \bigcup_{\alpha < \kappa} I_\alpha$ and $p \neq q \in S^I(M)$. Then, there is some $\alpha_0 < \kappa$ such that $p^{I_{\alpha_0}} \upharpoonright M_{\alpha_0} \neq q^{I_{\alpha_0}} \upharpoonright M_{\alpha_0}$.*

Proof: Let $M = \bigcup_{\alpha < \kappa} M_i$ and $p \neq q \in S(M)$, as above. Let $X = \langle x_i : i \in I \rangle$ and $Y = \langle y_i : i \in I \rangle$ realize p and q respectively. Since κ is measurable, there is some normal, κ-complete ultrafilter U on κ such that we get the following commuting and elementary diagram

$$V \xrightarrow{\;\;j\;\;} \mathcal{M}$$
$$i \searrow \qquad \nearrow \pi$$
$$\Pi V/U$$

where i is the ultrapower embedding, π is the Mostowski collapse, $crit\, j = \kappa$, and $^\kappa \mathcal{M} \subset \mathcal{M}$. Since V is the set-theoretic universe, we also have $\mathcal{M}, \Pi V/U \subset V$. Similarly, $j(I) = \bigcup_{\alpha < j(\kappa)} I'_\alpha$ and $I'_\kappa = \bigcup_{\alpha < \kappa} j(I_\alpha)$. Set $X'_\alpha = \langle x_i : i \in I'_\alpha \rangle$ and $Y'_\alpha = \langle y_i : i \in I'_\alpha \rangle$.

By elementarity, we have that $j(p) \neq j(q) \in (S^{j(I)}(j(M)))^{\mathcal{M}}$ and $j(M) = \bigcup_{\alpha < j(\kappa)} M'_\alpha$, where $\langle M'_\alpha : \alpha < j(\kappa) \rangle = j(\langle M_\alpha : \alpha < \kappa \rangle)$ and, for $\alpha < \kappa$, $M'_\alpha = j(M_\alpha)$. $\langle M'_\alpha : \alpha < j(\kappa) \rangle$ is continuous, so $M'_\kappa = \bigcup_{\alpha < \kappa} M'_\alpha = \bigcup_{\alpha < \kappa} j(M_\alpha)$.

A priori, all that is known is that $j(M) \in j(K)$, which $\mathcal{M}$ thinks is an AEC. In fact, $j(K) = K^{\mathcal{M}}$; this follows from Theorem 4.2.3 since, for any $N \in K$, $i(N) = \Pi N/U \in K$. Since π is an isomorphism, $\pi \circ i(N) = j(N) \in K$.

For any $N \in K$, we note that $i^{\prime\prime}N \in K$ is isomorphic to N and has universe $\{[\alpha \to n]_U : n \in |N|\}$, so $i^{\prime\prime}N \prec i(N) = \Pi N/U$ by the above. So $i^{\prime\prime}M_\alpha \prec i(M_\alpha)$ for every $\alpha < \kappa$. Thus $i^{\prime\prime}M_\alpha \prec i(M_\beta)$ for every $\beta \geq \alpha$ and, taking a union over the $\beta < \kappa$, $i^{\prime\prime}M_\alpha \prec \bigcup_{\beta < \kappa} i(M_\beta)$. Now taking a union over $\alpha < \kappa$, $\bigcup_{\alpha < \kappa} i^{\prime\prime}M_\alpha = i^{\prime\prime}M \prec \bigcup_{\beta < \kappa} i(M_\beta)$. Applying π to both sides yields

$$j^{\prime\prime}M = (\pi \circ i)^{\prime\prime}M \prec \pi(\bigcup_{\beta < \kappa} i(M_\beta)) \prec \bigcup_{\beta < \kappa} \pi \circ i(M_\beta) = M'_\kappa$$

Since $M'_\kappa \prec j(M)$, which is the domain for $j(p)$ and $j(q)$, we have that $j(p) \upharpoonright j^{\prime\prime}M$ and $j(q) \upharpoonright j^{\prime\prime}M$ are defined. Similarly, $j^{\prime\prime}I \subset I'_\kappa$ so $j^{\prime\prime}X \subset X'_\kappa$ and $j^{\prime\prime}Y \subset Y'_\kappa$.

We wish to show $j(p)^{I'_\kappa} \upharpoonright M'_\kappa$ and $j(q)^{I'_\kappa} \upharpoonright M'_\kappa$ are different. So we compute

$$
\begin{aligned}
i(p) &= tp(i(X)/i(M)) \\
&= tp((\Pi X/U)/(\Pi M/U))
\end{aligned}
$$

and $i(q) = tp((\Pi Y/U)/(\Pi M/U))$. However, we know that $tp(X/M) \neq tp(Y/M)$. So $tp(i^{\prime\prime}X/i^{\prime\prime}M) \neq tp(i^{\prime\prime}Y/i^{\prime\prime}M)$. Since $i^{\prime\prime}M \prec \Pi M/U$, $i^{\prime\prime}I \subset \Pi I/U$ and non-equality of types goes up, we have that $i(p)^{i^{\prime\prime}I} \upharpoonright i^{\prime\prime}M \neq i(q)^{i^{\prime\prime}I} \upharpoonright i^{\prime\prime}M$. Applying our isomorphism π, we get

$$
\begin{aligned}
\pi(i(p)^{i^{\prime\prime}I} \upharpoonright i^{\prime\prime}M) &\neq \pi(i(q)^{i^{\prime\prime}I} \upharpoonright i^{\prime\prime}M) \\
\pi \circ i(p)^{(\pi \circ i)^{\prime\prime}I} \upharpoonright (\pi \circ i)^{\prime\prime}M &\neq \pi \circ i(q)^{(\pi \circ i)^{\prime\prime}I} \upharpoonright (\pi \circ i)^{\prime\prime}M \\
j(p)^{j^{\prime\prime}I} \upharpoonright j^{\prime\prime}M &\neq j(q)^{j^{\prime\prime}I} \upharpoonright j^{\prime\prime}M
\end{aligned}
$$

as desired. Since $j^{\prime\prime}M \prec M'_\kappa$ and $j^{\prime\prime}I \subset I'_\kappa$, we have that $j(p)^{I'_\kappa} \upharpoonright M'_\kappa \neq j(q)^{I'_\kappa} \upharpoonright M'_\kappa$.

So far, we have argued completely in V. However, since equality of types is existentially witnessed and a

witness in $\mathcal{M}$ would also be a witness in V, this holds true in $\mathcal{M}$ as well. So, we get the following

$$\mathcal{M} \models j(p)^{I'_\kappa} \upharpoonright M'_\kappa \neq j(q)^{I'_\kappa} \upharpoonright M'_\kappa$$

$$\mathcal{M} \models \exists \alpha < j(\kappa) \text{ st for } N = \text{the } \alpha\text{th member of } j(\langle M_\beta : \beta < \kappa \rangle) \text{ and } J = \text{the } \alpha\text{th member of}$$
$$j(\langle I_\beta : \beta < \kappa \rangle), j(p)^J \upharpoonright N \neq j(q)^J \upharpoonright N$$

$$V \models \exists \alpha < \kappa \text{ st for } N = \text{the } \alpha\text{th member of } \langle M_\beta : \beta < \kappa \rangle \text{ and } J = \text{the } \alpha\text{th member of}$$
$$\langle I_\beta : \beta < \kappa \rangle, p^J \upharpoonright N \neq q^J \upharpoonright N$$

$$V \models \exists \alpha < \kappa \text{ st} p^{I_\alpha} \upharpoonright M_\alpha \neq q^{I_\alpha} \upharpoonright M_\alpha$$

Since V is the universe, there is some $\alpha_0 < \kappa$ such that $p^{I_{\alpha_0}} \upharpoonright M_{\alpha_0} \neq q^{I_{\alpha_0}} \upharpoonright M_{\alpha_0}$.   †

Corollary 4.3.3. K *is fully* $(< \lambda, \lambda)$*-tame and fully* $(< \lambda, \lambda)$*-type short whenever* $cf \lambda = \kappa$ *and* $\lambda > LS(K)$.

Finally, we wish to weaken the assumptions on the theorems above to remove the use of the monster model. Note that, because, in these contexts, we can always take an ultrapower and $M \precnsim \Pi M/U$ for any $M \in K$ at least the size of the completeness of the ultrafilter, we already have no maximal models. So, in particular, we remove the assumptions of amalgamation and joint embedding. The loss of amalgamation is particularly worrisome because it is used to prove that $\sim_{AT}$ is an equivalence relation and we only have that $\sim$ is a *non-trivial* transitive closure of $\sim_{AT}$. Also, we now use the complete strength of the closure theorem for ultraproducts of AECs.

Theorem 4.3.4. *Suppose* K *is an AEC essentially below* κ *measurable.* K *is fully* $(< \lambda, \lambda)$ *tame and fully* $(< \lambda, \lambda)$ *type short for* $\lambda > LS(K)$ *with* $cf \lambda = \kappa$.

Proof: For ease, we only show tameness. Type shortness follows similarly, but would add extra notation to an already notation heavy proof. Let $M \in K_\lambda$ and let $p, q \in S(M)$ such that $p \upharpoonright N = q \upharpoonright N$ for all $N \in P^*_\lambda M$. Find a, b, N^0, N^1 such that $p = tp(a/M, N^0)$ and $q = tp(b/M, N^1)$. Then we can find resolutions $\langle M_i, N^0_i, N^1_i \in K_{<\lambda} : i < \kappa \rangle$ of M, N^0, and N^1 respectively such that $a \in |N^0_0|$ and $b \in |N^1_0|$. Then we know that, for all $i < \kappa$, $(a, M_i, N^0_i) \sim (b, M_i, N^1_i)$. Since $\sim$ is the transitive closure of $\sim_{AT}$, for every $i < \kappa$, there is some $n_i < \omega$ such that, for all $\ell \leq n_i$, we have $N_i^{\frac{\ell}{n_i}}$ and a^i_ℓ such that $a^i_0 = a$, $a^i_{n_i} = b$, every $a^i_\ell \in |N_i^{\frac{\ell}{n_i}}|$, and

$$(a, M_i, N^0_i) \sim_{AT} (a^i_1, M_i, N_i^{\frac{1}{n_i}}) \sim_{AT} \cdots \sim_{AT} (a^i_{n_i-1}, M_i, N_i^{\frac{n_i-1}{n_i}}) \sim_{AT} (b, M_i, N^1_i)$$

Since there are only countably many choices for n_i and $cf \kappa > \omega$, there is some n that occurs cofinally often; WLOG, we may thin our sequence and assume $n_i = n$ for all $i < \kappa$. In particular, note that Theorem 4.2.3.5 does not require continuity. Now, by the definition of $\sim_{AT}$, for all $i < \kappa$ and $\ell < n$, there is some $N^*_{i,l} \succ N_i^{\frac{\ell+1}{n}}$ and $f_{i,\ell} : N_i^{\frac{\ell}{n}} \to N^*_{i,\ell}$ such that

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commutes and $f_{i,\ell}(a^i_\ell) = a^i_{\ell+1}$. Looking across all $\ell < n$, we get the following commuting diagram

$$
\begin{array}{ccccccc}
N^*_{i,1} & \cdots & N^*_{i,n-2} & \\
\uparrow {\scriptstyle f_{i,1}} & & \uparrow & \\
N^*_{i,0} \longleftarrow N^{\frac{1}{n}}_i & \cdots & N^{\frac{n-1}{n}}_i \xrightarrow{f_{i,n-1}} N^*_{i,n-1} \\
\quad {\scriptstyle f_{i,0}} & & \\
N^0_i \longleftarrow M_i \longrightarrow N^1_i &
\end{array}
$$

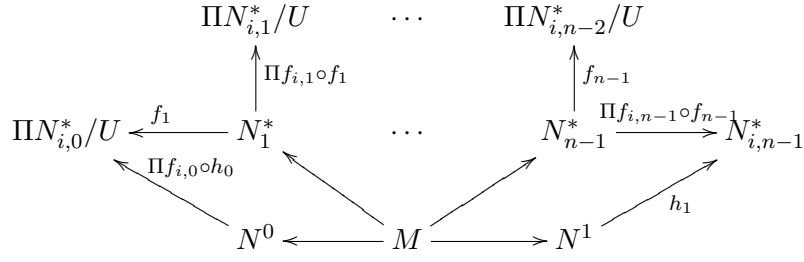so $f_{i,\ell}(a^i_\ell) = a^i_{\ell+1}$ for all $i < \kappa$ and $\ell < n$.

Let U be some κ-complete ultrafilter on κ. Now we take the ultraproduct of the above diagrams. Recall that we use Πf_i to denote the average of maps f_i; see the discussion after the proof of Theorem 4.2.3.

$$
\begin{array}{ccccccc}
\Pi N^*_{i,1}/U & \cdots & \Pi N^*_{i,n-2}/U & \\
\uparrow {\scriptstyle \Pi f_{i,1}} & & \uparrow & \\
\Pi N^*_{i,0}/U \longleftarrow \Pi N^{\frac{1}{n}}_i/U & \cdots & \Pi N^{\frac{n-1}{n}}_i/U \xrightarrow{\Pi f_{i,n-1}} \Pi N^*_{i,n-1}/U \\
\quad {\scriptstyle \Pi f_{i,0}} & & \\
\Pi N^0_i/U \longleftarrow \Pi M_i/U \longrightarrow \Pi N^1_i/U &
\end{array}
$$

Also by our hypotheses, if we take the function $h : M \to \Pi M_i/U$ given by $h(m) = [i \to m]_U$, then this is a K embedding. Note that, although the function $i \to m$ is not well-defined for all i, by U's κ completeness, it is defined on a measure one set, so the h is still well-defined. We can similarly define $h_0 : N_0 \to \Pi N^0_i/U$ and $h_1 : N_1 \to \Pi N^1_i/U$. Note that, for all $m \in M$, $h(m) = h_0(m) = h_1(m)$. These allow us to construct the following commutative diagram

$$
\begin{array}{ccccccc}
\Pi N^*_{i,1}/U & \cdots & \Pi N^*_{i,n-2}/U \\
\uparrow {\scriptstyle \Pi f_{i,1}} & & \uparrow \\
\Pi N^*_{i,0}/U \longleftarrow \Pi N^{\frac{1}{n}}_i/U & \cdots & \Pi N^{\frac{n-1}{n}}_i/U \xrightarrow{\Pi f_{i,n-1}} \Pi N^*_{i,n-1}/U \\
\uparrow {\scriptstyle \Pi f_{i,0}} & & \uparrow \\
\Pi N^0_i/U & \Pi M_i/U & \Pi N^1_i/U \\
{\scriptstyle h_0} & {\scriptstyle h} & {\scriptstyle h_1} \\
N^0 & N^1 \\
& M &
\end{array}
$$

This is essentially the diagram that we want, but we have to do some renaming to get it into the desired form. For each $1 \le \ell < n$, set $\mathbf{a}_\ell = [i \mapsto a^i_\ell]_U$ and $\bar{N}^*_\ell \prec \Pi N^{\frac{\ell}{n}}_i/U$ of size $\lambda + \ell(a)$ containing $\mathbf{a}_\ell$ and $h(M)$. Then find some $L(K)$ isomorphism f_ℓ that contains h with range $\bar{N}^*_\ell$ and N^*_ℓ such that $f_\ell : N^*_\ell \cong \bar{N}^*_\ell$. Set $a_\ell = f^{-1}_\ell(\mathbf{a}_\ell) \in |N^*_\ell|$. This gives us the diagram

$$\Pi N_{i,1}^* / U \qquad \cdots \qquad \Pi N_{i,n-2}^* / U$$

$$\Big\uparrow {\scriptstyle \Pi f_{i,1}\circ f_1} \qquad\qquad\qquad \Big\uparrow {\scriptstyle f_{n-1}}$$

$$\Pi N_{i,0}^*/U \xleftarrow{\ f_1\ } N_1^* \qquad \cdots \qquad N_{n-1}^* \xrightarrow{\ \Pi f_{i,n-1}\circ f_{n-1}\ } N_{i,n-1}^*$$

$$\nwarrow {\scriptstyle \Pi f_{i,0}\circ h_0} \qquad\qquad\qquad\qquad\qquad\qquad \nearrow {\scriptstyle h_1}$$

$$N^0 \longleftarrow M \longrightarrow N^1$$

This witnesses that

$$(a, M, N^0) \sim_{AT} (a_1, M, N_1^*) \sim_{AT} \cdots \sim_{AT} (a_{n-1}, M, N_{n-1}^*) \sim_{AT} (b, M, N^1)$$

So $(a, M, N^0) \sim (b, M, N^1)$ and $p = q$. $\hspace{3cm}$ †

4.4 Weakly Compact

In this section, we establish a number of downward reflection principles using indescribable cardinals. Tameness follows because it is a downward reflection of type inequality, but these principles apply to many other AEC properties as well.

Definition 4.4.1 (Indescribable Cardinals, [Kan08].1.6)**.**

1. *For $m, n < \omega$, a cardinal κ is Π_n^m-indescribable iff for any $R \subset V_\kappa$ and Π_n^m-statement ϕ in the language of $\{\in, R\}$, if $\langle V_\kappa, \in, R\rangle \vDash \phi$, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \vDash \phi$*

2. *κ is totally indescribable iff κ is Π_n^m-indescribable for all $n, m < \omega$.*

Although the indescribability definition is stated in terms of a single $R \subset V_\kappa$, a simple coding argument shows that it is equivalent to allow finitely many $R_0, \ldots, R_n \subset V_\kappa$ in the expanded language.

Remark 4.4.2 ([Kan08])**.** *An uncountable cardinal κ is weakly compact iff κ is Π_1^1 indescribable. Another definition is that any κ sized set of sentences from $L_{\kappa,\kappa}$ is consistent iff all of its $< \kappa$ sized subsets are. For context, if κ is measurable then it is Π_1^2 indescribable and, moreover, for any normal ultrafilter U on κ, $\{\alpha < \kappa : \alpha$ is totally indescribeable$\} \in U$.*

In the following lemma, we are going to code models of an AEC K with $LS(K) < \kappa$ as a subset of V_κ. In order to do this, we use the fact that there are two definable functions g and h so

- $g : \kappa \to P_\omega \kappa$ is a bijection such that, for all $\mu < \kappa$, we have that $g \restriction \mu : \mu \to P_\omega \mu$ is a bijection; and

- $h : \kappa \times {}^{LS(K)}2 \to V_\kappa$ is an injection such that, for all $2^{LS(K)} < \mu < \kappa$, we have that $h \restriction \mu : \mu \times {}^{LS(K)}2 \to V_\mu$ is an injection.

Lemma 4.4.3 (Coding Lemma)**.** *Suppose K is an AEC such that $LS(K) < \kappa$. There is $C_K \subset V_\kappa$ and Π_n^0 formulas $\phi(x), \psi(x, y), \sigma(x, y), \tau(x, y, z), \tau^+(x, y, z) \in L(\{\in, C_K\})$ such that for $\alpha \leq \kappa$ and $X, Y, f \subset V_\alpha$ and $a \in V_\alpha$, we have*

- $\langle V_\alpha, \in, C_K \cap V_\alpha \rangle \vDash \phi(X) \iff$

  C_K decodes from X an $L(K)$ structure M_X and $M_X \in K_{|\alpha|}$

- $\langle V_\alpha, \in, C_K \cap V_\alpha \rangle \vDash \psi(X, Y) \iff$

  C_K decodes from X and Y $L(K)$ structures M_X and M_Y, $M_X, M_Y \in K_{|\alpha|}$, and $M_X \prec_K M_Y$

- $\langle V_\alpha, \in, C_K \cap V_\alpha \rangle \vDash \sigma(X, a) \iff$

  C_K decodes from X $L(K)$ structures $M_X \in K_{|\alpha|}$ and $a \in M_X$

- $\langle V_\alpha, \in, C_K \cap V_\alpha \rangle \vDash \tau(X, Y, f) \iff$

  C_K decodes from X and Y $L(K)$ structures M_X and M_Y such that $M_X, M_Y \in K_{|\alpha|}$, and $f : M_X \to M_Y$

- $\langle V_\alpha, \in, C_K \cap V_\alpha \rangle \vDash \tau^+(X, Y, f) \iff$

  C_K decodes from X and Y $L(K)$ structures M_X and M_Y, $M_X, M_Y \in K_{|\alpha|}$, and $f : M_X \cong M_Y$

Proof: In the statement, we make reference to a "decoded structure," which we will explain. By Shelah's Presentation Theorem, we know $K = PC(T_1, \Gamma, L(K))$. Additionally, we can code $\prec_K$ as an AEC with Löwenheim-Skolem number $LS(K)$: set $K_\prec = \{(M, |M_0|) : M_0 \prec_K M\}$ and $\prec_{K_\prec} = \{((M, |M_0|), (N, |N_0|)) \in K_\prec \times K_\prec : M \prec_K N \text{ and } M_0 \prec_K N_0\}$. Then we have that $K_\prec = PC(T_2, \Gamma', L(K)')$. WLOG, we can assume that these objects are in $V_{(2^{LS(K)})^{+\omega}}$ and $L(T_1) = \langle R_i : i < |L(T_1)| \rangle$ and $L(T_2) = \langle S_i : i < |L(T_2)| \rangle$ are relational. Set $C_K = ((2^{LS(K)})^{+\omega}, L(K), \Gamma, T_1, \Gamma', T_2)$. Define a Π^0_n formula ϕ such that $\langle V_\kappa, \in, C_K \cap V_\alpha \rangle \models \phi(X)$ asserts all of the following

1. C_K is an ordered sextuple whose first element is an ordinal; this guarantees that the V_α that models it is above $2^{LS(K)}$ and, thus, can see the other elements.

2. X is in the range of h and $(h^{-1})''X$ is of the form $\{(i, f_i) : i < \alpha\}$. Set $C_i = \{j \in \alpha : f_j(i) = 1\}$.

3. $g''C_0$ should be a set of singletons; denote $\bigcup g''C_0$ by $|M_X|$.

4. $g''C_i$ should be a set of tuples whose length match the arity of R_i; denote this set $R_i^{M_X^+}$.

5. $M_X^+ = \langle |M_X|, R_i^{M_X^+} \rangle_{i < LS(K)}$ models T_1 and omits each $p \in \Gamma$.

6. Finally, M_X is the model $M_X^+ \restriction L(K)$.

Thus, $\langle V_\kappa, \in, C_K \cap V_\alpha \rangle \models \phi(X)$ iff $M_X \in K$ by Shelah's Presentation Theorem.

For $\psi(X, Y)$, we do a similar decoding process with T_2 and Γ'.

For $\sigma(X, a)$, we need to say that a is in the image of our decoding of C_0, which requires a quantifier over an element of X.

For $\tau^+(X, Y, f)$, we use ϕ to determine that X and Y are codes for elements of our PC class and then say that f is an isomorphism, which again just quantifies over elements of our models and L, all of which we have given.

For $\tau(X, Y, f)$, we have a definable way to talk about the image of X under f and combine ψ and τ^+ to say that f is an isomorphism between X and its image and that X's image is a $\prec_K$ submodel of Y. †

Now we are ready to begin proving theorems from this coding.

Theorem 4.4.4 (Tameness Down for Π_1^1). *Suppose K is an AEC such that $LS(K) < \kappa$ with κ-AP and κ being Π_1^1-indescribable. Then K is $(<\kappa, \kappa)$-tame for $<\kappa$-types.*

Proof: Let C_K be as in the Coding Lemma. Let $M \in K_\kappa$ and $p \neq q \in S(M)$. Then we have $p = tp(a/M, N_1)$ and $q = tp(b/M, N_2)$ for $M \prec N_1, N_2 \in K_\kappa$ and $a \in N_1$ and $b \in N_2$. WLOG, $|N_1| \cup |N_2| \subset V_\kappa$. Then

$$
\begin{aligned}
\models \quad & M, N_1, N_2 \in K_\kappa \\
\wedge \quad & M \prec N_1, N_2 \\
\wedge \quad & a \in N_1, b \in N_2 \wedge \forall N^* \in K_\kappa, \forall f_i : N_1 \to N^*(\forall m \in M(f_1(m) = f_2(m)) \to f_1(a) \neq f_2(b))
\end{aligned}
$$

Let $X, Y_1, Y_2 \subset V_\kappa$ code M, N_1, N_2, respectively, according to C_K. Then we rewrite the above as

$$
\begin{aligned}
\langle V_\kappa, \in, C_K, X, Y_1, Y_2, \{a\}, \{b\} \rangle \models \quad & \phi(X) \wedge \phi(Y_1) \wedge \phi(Y_2) \wedge \psi(X, Y_1) \wedge \psi(X, Y_2) \wedge \\
\wedge \quad & \sigma(X_1, a) \wedge \sigma(X_2, b) \\
\wedge \quad & \forall Y^*, f_1, f_2 \subset V_\kappa[(\phi(Y^*) \wedge \tau(Y_1, Y^*, f_1) \wedge \tau(Y_2, Y^*, f_2) \wedge \\
& \wedge [\forall x \in V_\kappa \sigma(X, x) \to (f_1(x) = f_2(x))]) \to (f_1(a) \neq f_2(b))]
\end{aligned}
$$

Since everything is first-order except for the single universal quantifier over subsets of V_κ, this is a Π_1^1 statement. So it reflects down to some $\alpha < \kappa$. Since for this to happen, $\{a\} \cap V_\alpha$ and $\{b\} \cap V_\alpha$ must be nonempty, we must have $a, b < \alpha$. Let $X' = X \cap V_\alpha$, $Y_1' = Y_1 \cap V_\alpha$, and $Y_2' = Y_2 \cap V_\alpha$. Then we have that $tp(a/M_{X'}, N_1') = p \upharpoonright M_{X'}$ and $tp(b/M_{X'}, N_2') = q \upharpoonright M_{X'}$.

Claim: $p \upharpoonright M_{X'} \neq q \upharpoonright M_{X'}$

If not, then there is some $N^* \in K_{|\alpha|}$ and $f_i : N_i' \to N^*$ that witnesses this with $f_1(m) = f_2(m)$ for all $m \in M$ and $f_1(a) = f_2(b)$. However, WLOG, $|N^*| \subset \alpha$, so we can code N^* as $Y^* \subset V^\alpha$ according to C_K. Then $f_1, f_2 \subset V_\alpha$ and Y^*, f_1, f_2 serve as a counterexample for our downward reflection.

Thus, we have our $M_{X'} \in K_{<\kappa}$ such that p and q differ on their restriction to $M_{X'}$. †

Above, we assumed amalgamation to simplify the exposition. However, we could drop this assumption without difficulty by adding a (first-order) quantifier to see how many steps it might take to show p and q are equal.

A similar argument gives us a result for type shortness.

Theorem 4.4.5 (Tameness Down for Π_1^1). *Suppose K is an AEC such that $LS(K) < \kappa$ with κ-AP and κ being Π_1^1-indescribable. Then K is $(<\kappa, \kappa)$-type short over $<\kappa$-sized models.*

This method is not just useful for tameness and type shortness. It can be used to reflect many AEC properties down. Only the amount of indescribility required changes from property to property. For instance,

Theorem 4.4.6 (Unbounded Categoricity Down for Π_2^1). *Suppose K is an AEC such that $LS(K) < \kappa$ with κ being Π_2^1-indescribable. Then for every $\lambda < \kappa$, there is some $\lambda < \mu < \kappa$ such that K is μ-categorical.*

Proof: Let $\lambda < \kappa$. Code K by C_K. We want to find $\lambda < \mu < \kappa$ such that that K is μ-categorical. Since K is κ categorical,

$$\models \quad \forall M, N \in K_\kappa, \exists f : M \cong N$$

$$\langle V_\kappa, \in, C_K, \{\lambda^+\}\rangle \quad \models \quad \forall X, Y \subset V_\kappa \exists f \subset V_\kappa [\phi(X) \wedge \phi(Y) \rightarrow \tau^+(X, Y, f)] \wedge \exists x(x \in \{\lambda^+\})$$

Then this reflects down to some $\alpha < \kappa$. Since $V_\alpha \cap \{\lambda^+\}$ is not empty, we get that $\alpha > \lambda^+$, so $|\alpha| > \lambda$. Set $\mu = |\alpha|$ and let $M, N \in K_\mu$. WLOG, $|M|, |N| \subset \alpha$, so we can code these by X and Y, respectively. Then

$$\langle V_\alpha, \in, C_K \cap V_\alpha\rangle \models \phi(X) \wedge \phi(Y)$$

Since our statement of categoricity reflects down to α, there is some $f \in V_\alpha$ such that $f : M \cong N$. So K is μ categorical. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\dagger$

Remark 4.4.7. *Recalling what has been said about work on Shelah's Categoricity Conjecture, one may initially hope that this downward reflection might be massaged to make the downward reflection hold at a successor cardinal. However, this is unlikely, since successor (and singular limit) cardinals are necessarily first-order describable, so all we could guarantee of μ is that it is strongly inaccessible.*

We have many other theorems of this type:

Theorem 4.4.8 (Unbounded (Disjoint) Amalgamation Down for Π_2^1). *Suppose K is an AEC such that $LS(K) < \kappa$ with κ (disjoint) amalgamation and κ being Π_2^1-indescribable. Then, for every $\lambda < \kappa$, there is some $\lambda < \mu < \kappa$ such that K has the μ (disjoint) amalgamation.*

Theorem 4.4.9 (Unbounded Uniqueness of Limit Models Down for Π_1^2). *Suppose K is an AEC such that $LS(K) < \kappa$ with κ being Π_1^2-indescribable. If K_κ has a unique limit model, then, for every $\lambda < \kappa$, there is some $\lambda < \mu < \kappa$ such that K_μ has a unique limit model.*

The general heuristic for determining how much indescribability is required to transfer a property of an AEC down is to look at the quantifiers needed to state this property and translate quantifiers over elements to Π^0 quantifiers; over models or embeddings to Π^1 quantifiers; and over sequences of models or embeddings to Π^2 quantifiers. Following this, sequences of sequences of models would require Π^3 quantifiers, but there seem to be no useful AEC properties requiring a quantifier of this sort.

4.5 Conclusion

In this section, we prove the consistency of Shelah's Eventual Categoricity Conjecture for Successors by combining our results with those of [GV06a] and [Sh394]. After doing so, we apply our results to other results in the literature.

Before we can apply the results of of [GV06a] and [Sh394], we must show that categoricity implies their hypotheses of no maximal models, joint embedding, and amalgamation. If K is the class of models of some $L_{\kappa,\omega}$ sentence, then this is done in [MaSh285].§1. We generalize these arguments to an AEC K with $LS(K) < \kappa$ by introducing the notion of universal closure as a generalization of existential closure.

Definition 4.5.1. $M \in K$ *is called universally closed iff given any* $N \prec M$ *and* $N' \succ N$, *both of size less than* κ, *if there is* $M^+ \succ M$ *and* $g : N' \to_N M^+$, *then there is* $f : N' \to_N M$.

We omit the parameter κ from the name because it will always be fixed and clear from context. Note that if there is an M^+ witnessing that M is not universally closed, then there is one of size $\|M\|$.

Recall that M is an amalgamation base when all M_1 and M_2 extending M can be amalgamated over M.

Lemma 4.5.2. *Suppose* K *is an AEC and* κ *is strongly compact such that* $LS(K) < \kappa$. *Then any universally closed* $M \in K_{\geq \kappa}$ *is an amalgamation base.*

Proof: Let M be universally closed and $M \prec M_1, M_2$. First, we show we can amalgamate every small approximation of this system. Let $N \prec M$ and $N_\ell \prec M_\ell$ such that $N \prec N_\ell$ for $\ell = 1, 2$ with $N, N_1, N_2 \in K_{<\kappa}$. Then M_ℓ is an extension of M such that N_ℓ can be embedded into it over N. Since M is universally closed, there is $f_\ell : N_\ell \to_N M$. Find $N_* \prec M$ of size $< \kappa$ such that $f_1(N_1), f_2(N_2) \prec N_*$. Then this is an amalgamation of N_1 and N_2 over N.

Now we will use our strongly compact cardinal. Set

$$X = \{\mathbf{N} = (N^{\mathbf{N}}, N_1^{\mathbf{N}}, N_2^{\mathbf{N}}) \in (K_{<\kappa})^3 : N^{\mathbf{N}} \prec M, N_\ell^{\mathbf{N}} \prec M_\ell, N^{\mathbf{N}} \prec N_\ell^{\mathbf{N}} \text{ for } \ell = 1, 2\}$$

For each $\mathbf{N} \in X$, the above paragraphs shows that there is an amalgam of this triple. Fix $f_\ell^{\mathbf{N}} : N_\ell^{\mathbf{N}} \to N_*^{\mathbf{N}}$ to witness this fact. For each $(A, B, C) \in [M]^{<\kappa} \times [M_1]^{<\kappa} \times [M_2]^{<\kappa}$, define

$$[(A, B, C)] := \{\mathbf{N} \in X : A \subset N^{\mathbf{N}}, B \subset N_1^{\mathbf{N}}, C \subset N_2^{\mathbf{N}}\}$$

These sets generate a κ-complete filter on X, so it can be extended to a κ-complete ultrafilter U. By Łoś' Theorem for AECs, since this ultrafilter is fine, we know that the ultrapower map h is a K-embedding, so

$$h : M \to \Pi N^{\mathbf{N}}/U$$
$$h_\ell : M_\ell \to \Pi N_\ell^{\mathbf{N}}/U \qquad \text{for } \ell = 1, 2$$

Since these maps have a uniform definition, they agree on their common domain M. Furthermore, we can average the $f_\ell^{\mathbf{N}}$ maps to get

$$\Pi f_\ell^{\mathbf{N}} : \Pi N_\ell^{N}/U \to \Pi N_*^{\mathbf{N}}/U$$

and the maps agree on $\Pi N^{\mathbf{N}}/U$ since each of the individual functions do. Then we can put these maps together to get the following commutative diagram that witnesses the amalgamation of M_1 and M_2 over M.

Now we use this result to derive the needed properties from categoricity. We focus on the case where K is categorical in λ of cofinality at least κ because it is simpler and suffices for our application. However, the methods of [MaSh285] can extend these results to categoricity in other cardinals. We use here the result of Solovay [Sol74] that $\mathrm{cf}\,\mu \geq \kappa$ implies $\mu^{<\kappa} = \mu$ when κ is strongly compact.

Proposition 4.5.3. *Suppose K is an AEC such that $LS(K) < \kappa$ strongly compact. If K is categorical in λ such that $\mathrm{cf}\,\lambda \geq \kappa$, then $K_{\geq\kappa}$ has amalgamation, joint embedding, and no maximal models.*

Proof: $K_{\geq\kappa}$ has no maximal models by Łoś' Theorem for AECs, since a model can be strictly embedded into its ultraproduct. This doesn't use categoricity and only needs κ to be measurable.

For joint mapping, we can use categoricity and no maximal models to get joint mapping below and at the categoricity cardinal. Above the categoricity cardinal, we use amalgamation and categoriciy. This relies only on the other properties and not directly on any large cardinals.

For amalgamation, we use the above result that universally closed models are amalgamation bases.

First, we show that a universally closed model exists in any cardinal μ of cofinality at least κ, which includes the categoricity cardinal. Let $M \in K_\mu$ and consider all possible isomorphism types of $N \prec N'$ from $K_{<\kappa}$ with $N \prec M$. There are at most $\mu^{<\kappa} \cdot 2^{<\kappa} = \mu$ many such types. We enumerate them (N_α, N'_α) for $\alpha < \mu$. Set $M = M_0$. Then for each $\alpha < \mu$, if there is some $M_\alpha^+ \succ M_\alpha$ of size μ such that there is $g : N'_\alpha \to_{N_\alpha} M_\alpha^+$ but no $f : N'_\alpha \to_{N_\alpha} M_\alpha$, then set $M_{\alpha+1} = M_\alpha^+$. Otherwise, $M_{\alpha+1} = M_\alpha$. At limit α, we take limits of the increasing chain. Set $M^* = \cup_{\alpha<\mu} M_\alpha \in K_\mu$.

Now we iterate this process κ many times: set $M^0 = M$, $M^{\alpha+1} = (M^\alpha)^*$, and $M^\alpha = \cup_{i<\beta} M^i$ for limit $\alpha \leq \kappa$. Then, M^κ is universally closed. By λ categoricity, this means that every model in K_λ is universally closed.

Second, we show that every model in $K_{>\lambda}$ is a universally closed. Let $M \in K_{>\lambda}$. Suppose that there are $N \prec N' \in K_{<\kappa}$ and $M^+ \succ M \succ N$ and $g : N' \to_N M^+$. Let $M' \prec M$ be of size λ and contain N. Then, by the above, M' is universally closed with $M^+ \succ M'$, so there is $f : N' \to_N M'$. Then $f : N' \to_N M$. Since N and N' were arbitrary, M is universally closed.

Third, we show that all models in $K_{\geq\kappa}$ are amalgamation bases and, thus, $K_{\geq\kappa}$ has the amalgamation property. Let $M \prec M_1, M_2$. If $M \in K_{\geq\lambda}$, then M is universally closed and, thus, an amalgamation base by Lemma 4.5.2. If not, then we can find some κ complete ultrafilter U and take an ultraproduct to get a proper extension
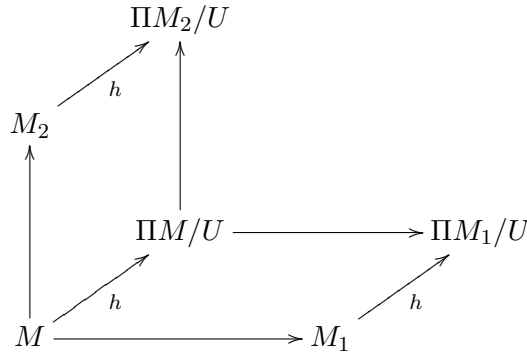
This is a larger triple of models that, if we could amalgamate it, would give us an amalgamation of M_1, M_2 over M. Then, we can continue to take ultrapowers of this triple, taking direct limits at unions,

until the base model has size at least λ. Then, by the above, it must be an amalgamation base, so we can amalgamate M_1 and M_2 over M.

Thus, all models in $K_{\geq\kappa}$ are amalgamation bases, so $K_{\geq\kappa}$ has the amalgamation property. †

Now that we have amalgamation, joint embedding, and no maximal models, we can generalize the result of [MaSh285] to all AECs essentially below a strongly compact.

Theorem 4.5.4. *Suppose κ is a strongly compact cardinal and K is an AEC essentially below κ. If K is categorical in some successor λ^+ greater than $\kappa^+ + LS(K)^+$, then it is categorical in all $\mu \geq \min\{\lambda^+, \beth_{(2^{Hanf(LS(K))})^+}\}$.*

Proof: By Theorem 4.2.5, K is $< (\kappa + LS(K)^+)$ tame, so it is $\kappa + LS(K)^+$ tame. Then, $K_{\geq\kappa}$ is an AEC with $LS(K_{\geq\kappa}) = \kappa$ that is κ-tame. Additionally, by Proposition 4.5.3, K has amalgamation, joint embedding, and no maximal models. Thus, by [GV06a].5.2, we know that K is categorical for every $\mu \geq \lambda^+$. Then K is definitely categorical in a successor above $\beth_{(2^{Hanf(LS(K))})^+}$. So, by [Sh394].9.5, it is categorical everywhere down to $\beth_{(2^{Hanf(LS(K))})^+}$. †

Note that the downward categoricity transfer result from [Sh394] does not use any tameness assumption. This result shows that given an AEC wtih amalgamation that is categorical in a successor cardinal λ above $\beth_{(2^{Hanf(LS(K))})^+}$, this AEC is also categorical in all cardinals in the interval $[\beth_{(2^{Hanf(LS(K))})^+}, \lambda]$.

Now we show that Shelah's Eventual Categoricity Conjecture for Successors follows from large cardinal assumptions:

Theorem 4.5.5. *If there are proper class many strongly compact cardinals, then Shelah's Eventual Categoricity Conjecture for Successors holds.*

Proof: Let λ be a cardinal and pick $\mu_\lambda = \min\{\mu^+ : \mu \geq \lambda$ and μ is strongly compact $\}$ Note that $\beth_{(2^{Hanf(\lambda)})^+} < \mu_\lambda$. If K is categorical in some successor μ above μ_λ, then Theorem 4.5.4 implies that K is categorical everywhere above μ_λ. †

While the hypothesis of this theorem seems very strong, we do note that [Jec06].20.22 and .24 show that the consistency of it follows from the existence of an extendible cardinal λ; in fact, V_λ is a model of the hypothesis.

Beyond the categoricity result, [MaSh285] introduces a very well behaved independence relation similar to the first-order notion of coheir. We generalize this in the next chapter and its uniqueness is established in Boney, Grossberg, Kolesnikov, and Vasey [BGKV]. Of particular note is that no large cardinal hypothesis is need, only the conclusions of Theorem 4.2.5 for a specific AEC.

Of particular interest in the proof of 4.5.5 is that we get, from the hypothesis of a proper class of strongly compact cardinals, the conclusion that *every* AEC with arbitrarily large models is eventually tame. Examining the ZFC counterexamples of [HaSh323] [BK09], the proven failure of tameness occurs at some small level bounded by $\aleph_\omega$. However, these classes have arbitrarily large models, so our results can apply. In particular, if there is a strongly compact cardinal, these classes exhibit the strange behavior of being $(\aleph_0, \aleph_k)$-tame very low, failing to be $(\aleph_k, \aleph_{k+1})$-tame, and then becoming $< \kappa$ tame at the strongly compact cardinal.

Turning to measurable cardinals, [KoSh362] derive amalgamation from categoricity and [Sh472] proves a downward categoricity transfer in $L_{\kappa,\omega}$. However, the papers do not use the specifics of $L_{\kappa,\omega}$ beyond that it is closed under κ complete ultralimits, see [KoSh362].1.7.1. The methods of Theorem 4.2.3 can be used to show closure under these ultralimits as well. Thus, we can extend their work to get the following results:

Theorem 4.5.6. *Suppose K is an AEC such that $LS(K) < \kappa$ measurable. If K is categorical in some $\lambda \geq \kappa$, then*

1. *$K_{[LS(K)+\kappa,\lambda)} = \{M \in K : LS(K) + \kappa \leq \|M\| < \lambda\}$ has the amalgamation property; and*

2. *if λ is also a successor above $\beth_{(2^{LS(K)})^+}$, then K is categorical in all μ with $\beth_{(2^{LS(K)})^+} \leq \mu \leq \lambda$.*

Beyond ultralimits, stronger large cardinals have more complicated constructions that witness their existence, such as extenders for strong cardinals [Jec06].20.28. Again, arguments similar to Theorem 4.2.3 will show closure under these constructions as well for AECs essentially below them.

4.6 Further work

As always, new answers lead to new questions.

In this paper, we have shown that the following statements follow from different large cardinals:

$(*)^-_\kappa$ Every AEC K with $LS(K) < \kappa$ is $(< \kappa, \kappa)$-tame.

$(*)_\kappa$ Every AEC K with $LS(K) < \kappa$ is $< \kappa$-tame.

$(*)$ Every AEC K with arbitrarily large models is $< \lambda$-tame in some $\lambda > LS(K)$.

We proved the same results for type shortness, but we focus this discussion on tameness because more is known.

A natural investigation is into these properties on their own. Can they hold at small cardinal? If so, do they have large cardinal strength?

A basic first result is that none of these properties can hold at $\aleph_k$ for $k < \omega$. This follows from the Hart-Shelah examples [HaSh323] [BK09].

A second result is that $(*)^-_\kappa$ for κ regular and not weakly compact implies $V \neq L$. To see this, first recall that Baldwin and Shelah [BlSh862] construct an AEC that is not $(< \kappa, \kappa)$ tame from an almost free, non-free, non-Whitehead group of size κ. In L, such a group is known to exists at precisely the non-weakly compact, regular cardinals; see Eklof and Mekler's book [EM02]. Combining these two facts, we have our proof. The construction in Eklof and Mekler has two main steps:

- non-reflecting stationary sets are used to construct almost free, non-free groups of every cardinality; and

- weak diamond on every stationary set is used to inductively show that all Whitehead groups are free.

The non-reflecting stationary sets suggest a natural tension with the compactness of the cardinals used in this paper. However, also being non-Whitehead seems to be a crucial part of the Baldwin and Shelah construction. It is not currently known if non-tameness follows just from almost free, non-freeness nor what the precise conditions for the existence of an almost free, non-free, non Whitehead group are.

A potential first step in achieving $(*)_\kappa^-$ and the other properties at small cardinals is the work of Magidor and Shelah in [MaSh204]. Starting from ω many supercompact cardinals, they construct

1. a model where every $\aleph_{\omega^2+1}$-free group is $\aleph_{\omega^2+2}$-free; and

2. a model where every κ-free group is free for $\kappa = \min\{\lambda \in \text{CARD} : \lambda = \aleph_\lambda\}$.

In the first model, there is no known candidate for a counterexample to $(*)_{\aleph_{\omega^2+2}}^-$ and, in the second, there is no known candidate for a counterexample to $(*)_{\aleph_\kappa}$. Further investigation will be needed to determine if these properties hold or if there are more non-tame AECs.

Nonforking in Short and Tame Abstract Elementary Classes

5.1 Introduction

In this chapter, we generalize the nonforking relation of coheir from stable first-order theories to type short and tame AECs without the weak κ-order property.

As an intermediary, between the first-order case (described in Pillay [Pil83]) and the AEC case, we review the results of Makkai and Shelah from [MaSh285]. They analyzed categoricity in $L_{\kappa,\omega}$ when κ is strongly compact. This puts them in the context of the last chapter and they directly showed that Galois types correspond to consistent sets of formulas from a fragment of $L_{\kappa,\kappa}$. Armed with this syntax, they define a nonforking relation by $p \in S(B)$ does not fork over A iff every $\phi(\mathbf{x}, \mathbf{b}) \in p$ is satisfied by an element of A. Note that $\phi(\mathbf{x}, \mathbf{b})$ comes from $L_{\kappa,\kappa}$, so this relation trades the finite satisfiability of first-order coheir of for $< \kappa$-satisfiability. This chapter, which is joint work with Rami Grossberg, is an extension and generalization of these results.

Recall the following definition from Chapter II.

Definition 5.1.1. *Let $M_0 \prec N$ be models and A be a set. We say that $tp(A/N)$ does not fork over M_0, written $A \underset{M_0}{\downarrow} N$, iff for all small $a \in A$ and all small $N^- \prec N$, we have that $tp(a/N^-)$ is realized in M_0.*

Thus a type does not fork over a base model iff all small approximations to it, both in length and domain, are realized in the base model. This definition is a relative of the finite satisfiability–also known as coheir–notion of forking that is extensively studied in stable theories.

We deal with a more general situation than [MaSh285], as our class is assumed to be an AEC that doesn't have a specific logic that axiomatizes it. Instead of categoricity, we assume a stability-like lack of a weak κ-order property and, instead of their large cardinal assumption, we assume the (weaker) model theoretic properties of tameness and type-shortness.

Unfortunately, there is no free lunch and we pay for this luxury. Our payment is essentially in assuming tameness and type-shortness. As was shown in the last chapter, these assumptions are corollaries of certain large cardinal axioms, including the one assumed by Makkai and Shelah. It seems to be plausible that tameness and type shortness will be derived in the future from categoricity above a certain Hanf number that depends only on $LS(K)$, see Conjecture [GV06a].1.5.

In this paper, we introduce a notion that, like the one from [MaSh285], is an analogue of the first order notion of coheir. One of our main results is that, given certain model theoretic assumptions, this notion is in fact an independence notion.

Theorem (5.3.1). *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If there is some $\kappa > LS(K)$ such that that*

1. *K is fully $< \kappa$-tame;*

2. *K is fully $< \kappa$-type short;*

3. *K doesn't have an order property; and*

4. *$\downarrow$ satisfies existence and extension,*

   then $\downarrow$ is an independence relation.

In a meeting at AIM that was dedicated to Classification Theory for AECs, there was a problem session moderated by Andres Villaveces [Vi06]. John Baldwin asked "Does Shelah's rank satisfy the Lascar inequalities, or is there another rank which does?" (in the context of Shelah's excellent classes). Theorem 5.5.7 provides an affirmative answer (for a much wider context). Another question asked by Baldwin and Grossberg at that meeting was "What is superstability for AECs?". While several approximations were offered by various authors, Theorem 5.5.9 provides an approximation to this question.

Section 5.2 gives a fine analysis of when parameterized versions of the axioms hold about our forking relation. Section 5.3 gives the global assumptions that make our forking relation an independence relation. Section 5.4 introduces a notion that generalizes coheir and deduces local character of our forking from this and categoricity. Section 5.5 introduces a U rank and shows that it is well behaved. Section 5.6 continues the study of large cardinals from the last chapter and shows that large cardinal assumptions simplify many of the previous sections.

The following hypotheses are used throughout this chapter.

Hypothesis 5.1.2. *Assume that K has no maximal models and satisfies the λ-joint embedding and λ-amalgamation properties for all $\lambda \geq LS(K)$.*

Fix a cardinal $\kappa > LS(K)$. All subsequent uses of κ will refer to this fixed cardinal, until Section 5.6. If we refer to a model, tuple, or type as 'small,' then we mean its size is $< \kappa$, its length is of size $< \kappa$, or both its domain and its length are small.

Although we do not use it in this chapter, we explain the changes that must be made if we don't work inside of a monster model but still assume amalgamation. In that case, the definition of the type of A over N must be augmented by a model containing both; that is, some $\widehat{M} \in K$ such that $A \subset |\widehat{M}|$ and $N \prec M$. We denote this type $tp(A/N, \widehat{M})$. Similarly, we must add this fourth input to the nonforking relation that contains all other parameters. Then $A \underset{M_0}{\overset{\widehat{M}}{\downarrow}} N$ iff $M_0 \prec N \prec \widehat{M}$ and $A \subset |\widehat{M}|$ and all of the small approximations to the type of A over N *as computed in* $\widehat{M}$. The properties are expanded similarly with added monotonicity for changing the ambient model $\widehat{M}$ and the allowance that new models that are found by properties such as existence or symmetry might exist in a larger big model $\widehat{N}$. All theorems proved in this paper about nonforking only require amalgamation, although some of the results referenced make use of the full power of the monster model.

We end this section with an easy exercise in the definition of nonforking that says that A and N must be disjoint outside of M_0.

Proposition 5.1.3. *If we have $A \underset{M_0}{\downarrow} N$, then $A \cap |N| \subset |M_0|$.*

Proof: Let $x \in A \cap |N|$. Since N is a model, we can find a small $N^- \prec N$ that contains x. Then, by the definition of nonforking, $tp(x/N^-)$ must be realized in M_0. But since $x \in |N^-|$, this type is algebraic so the only thing that can realize it is x. Thus, $x \in |M_0|$. †

5.2 Connecting Existence, Symmetry and Uniqueness

In this section, we investigate what AEC properties cause the axioms of our independence relation to hold. The relations are summarized in the proposition below.

Proposition 5.2.1. *Suppose that K doesn't have the weak κ-order property and is $(<\kappa, \lambda + \chi)$-type short for θ-sized domains and $(<\kappa, \theta)$-tame for $<\kappa$ length types. Then*

1. *$(E)_{(\chi,\theta,\lambda)}$ implies $(S)_{(\lambda,\theta,\chi)}$.*

2. *$(S)_{(<\kappa,\theta,<\kappa)}$ implies $(U)_{(\lambda,\theta,\chi)}$.*

This proposition and the lemma used to prove it below rely on an order property.

Definition 5.2.2. *K has the* weak κ-order property *iff there are lengths $\alpha, \beta < \kappa$, a model $M \in K_{<\kappa}$, and types $p \neq q \in S^{\alpha+\beta}(M)$ such that there are sequences $\langle a_i \in {}^{\alpha}\mathfrak{C} : i < \kappa \rangle$ and $\langle b_i \in {}^{\beta}\mathfrak{C} : i < \kappa \rangle$ such that, for all $i, j < \kappa$,*

$$i \leq j \implies tp(a_i b_j / M) = p$$
$$i > j \implies tp(a_i b_j / M) = q$$

This order property is a generalization of the first order version to our context of Galois types and infinite sequences. This is one of many order properties proposed for the AEC context (we introduce another one in Section 5.4) and is similar to 1-stability that is studied by Shelah in [Sh1019] in the context of $L_{\theta,\theta}$ theories where θ is strongly compact. The adjective 'weak' is in comparison to the $(<\kappa, \kappa)$-order property in Shelah [Sh394]. The key difference is that [Sh394] requires the existence of ordered sequences of *any* length, while we only require a sequence of length κ. We discuss the implications of the weak κ-order property property in the next section. For now, we use it to prove the following result, similar to one in [MaSh285].

Lemma 5.2.3. *Suppose K is an AEC that is $(<\kappa, \theta)$-tame for $<\kappa$ length types and doesn't have the weak κ-order property. Let $M_0 \prec M, N$ such that $\|M_0\| = \theta$ and let $a, b, a' \in \mathfrak{C}$ such that $\ell(a) = \ell(a') < \kappa$, $\ell(b) < \kappa$, $b \in N$, and $a' \in M$. If*

$$tp(a/M_0) = tp(a'/M_0) \text{ and } a \underset{M_0}{\downarrow} N, \text{ and } b \underset{M_0}{\downarrow} M$$

then $tp(ab/M_0) = tp(a'b/M_0)$.

Proof: Assume for contradiction that $tp(ab/M_0) \neq tp(a'b/M_0)$. We will build sequences that witness the weak κ-order property. By tameness, there is some $M^- \prec M_0$ of size $< \kappa$ such that $tp(ab/M^-) \neq tp(a'b/M^-)$. Set $p = tp(ab/M^-)$ and $q = tp(a'b/M^-)$. We will construct two sequences $\langle a_i \in {}^{\ell(a)}M_0 : i < \kappa \rangle$ and $\langle b_i \in {}^{\ell(b)}M_0 : i < \kappa \rangle$ by induction. We will have, for all $i < \kappa$

1. $a_i b \vDash p$;

2. $a_i b_j \vDash q$ for all $j < i$;

3. $a b_i \vDash q$; and

4. $a_i b_j \vDash p$ for all $j \geq i$.

Note that, since $b_i \in M_0$, (3) is equivalent to $a' b_i \vDash q$.

This is enough: (2) and (4) are the properties necessary to witness the weak κ-order property.

Construction: Let $i < \kappa$ and suppose that we have constructed the sequence for all $j < i$. Let $N^+ \prec N$ of size $< \kappa$ contain b, M^-, and $\{b_j : j < i\}$ Because $a' \underset{M_0}{\downarrow} N$, there is some $a_i \in M_0$ that realizes $tp(a/N^+)$. This is witnessed by $f \in Aut_{N^+}\mathfrak{C}$ with $f(a) = a_i$.

Claim: (1) and (2) hold.

f fixes M^- and b, so it witnesses that $tp(ab/M^-) = tp(a_ib/M^-)$. Similarly, it fixes b_j for $j < i$, so it witnesses $q = tp(ab_j/M^-) = tp(a_ib_j/M^-)$. $\dagger_{Claim}$ Similarly, pick $M^+ \prec M$ of size $< \kappa$ to contain M^-, a', and $\{a_j : j \leq i\}$. Because $b \underset{M_0}{\downarrow} M'$, there is $b_i \in M_0$ that realizes $tp(b/M^+)$. As above, (3) and (4) hold. $\hfill \dagger$

Now we are ready to prove our theorems regarding when the properties of $\downarrow$ hold. The first four properties always hold from the definition of nonforking.

Theorem 5.2.4. *If K is an AEC with $LS(K) < \kappa \leq \lambda$, then $\downarrow$ satisfies (I), (M), (T), and $(C)_{<\kappa}$.*

To get the other properties, we have to rely on some degree of tameness, type shortness, no weak order property, and the property (E).

Proof of Proposition 5.2.1:

1. Suppose $(E)_{(\chi,\theta,\lambda)}$ holds. Let $A_2 \underset{M_0}{\downarrow} M_1$ and $A_1 \subset |M_1|$ with $|A_2| = \lambda$, $\|M_0\| = \theta$, and $|A_1| = \chi$. Let M_2 contain A_2 and M_0 be of size λ. By $(E)_{(\chi,\theta,\lambda)}$, there is some A_1' such that $tp(A_1/M_0) = tp(A_1'/M_0)$ and $A_1' \underset{M_0}{\downarrow} M_2$. It will be enough to show that $tp(A_1A_2/M_0) = tp(A_1'A_2/M_0)$. By $(< \kappa, \lambda + \chi)$-type shortness over θ-sized domains, it is enough to show that, for all $a_2 \in A_2$ and corresponding $a_1 \in A_1$ and $a_1' \in A_1'$ of length $< \kappa$, we have $tp(a_1a_2/M_0) = tp(a_1'a_2/M_0)$. By (M), we have that $a_1' \underset{M_0}{\downarrow} M_2$ and $a_2 \underset{M_0}{\downarrow} M_1$, so this follows by Lemma 5.2.3 above.

   Now that we have shown the type equality, let $f \in Aut_{M_0}\mathfrak{C}$ such that $f(A_1A_2) = A_1'A_2$. Applying f to $A_1' \underset{M_0}{\downarrow} M_2$, we get that $A_1 \underset{M_0}{\downarrow} f(M_2)$ and $A_2 = f(A_2) \subset f(M_2)$, as desired.

2. Suppose $(S)_{(<\kappa,\theta,<\kappa)}$. Let A and A' be sets of size λ and $M_0 \prec N_0$ of size θ and χ, respectively, so that

$$tp(A/M_0) = tp(A'/M_0) \text{ and } A \underset{M_0}{\downarrow} N \text{ and } A' \underset{M_0}{\downarrow} N$$

   As above, it is enough to show that $tp(AN/M_0) = tp(A'N/M_0)$. By type shortness, it is enough to show this for every $n \in N$ and corresponding $a \in A$ and $a' \in A'$ of lengths less than κ. By (M), we know that $a \underset{M_0}{\downarrow} N$ and $a' \underset{M_0}{\downarrow} N$. By applying $(S)_{(<\kappa,\theta,<\kappa)}$ to the former, there is $N_a \succ M_0$ containing a such that $n \underset{M_0}{\downarrow} N_a$. As above, Lemma 5.2.3 gives us the desired conclusion. $\hfill \dagger$

5.3 The main theorem

We now state the ideal conditions under which our nonforking works.

Theorem 5.3.1. *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If there is some $\kappa > LS(K)$ such that*

1. *K is fully $< \kappa$-tame;*

2. *K is fully $< \kappa$-type short;*

3. *K doesn't have the weak κ-order property; and*

4. *$\mathord{\underset{\smile}{\;}}$ satisfies (E)*

then $\mathord{\underset{\smile}{\;}}$ is an independence relation.

Proof: First, by Theorem 5.2.4, $\mathord{\underset{\smile}{\;}}$ always satisfies (I), (M), (T), and $(C)_{<\kappa}$. Second, (E) is part of the hypothesis. Third, by the other parts of the hypothesis, we can use Proposition 5.2.1. Let χ, θ, and λ be cardinals. We know that $(E)_{(\chi,\theta,\lambda)}$ holds, so $(S)_{(\lambda,\theta,\chi)}$ holds. From this, we also know that $(S)_{(<\kappa,\theta,<\kappa)}$ holds. Thus, $(U)_{(\lambda,\theta,\chi)}$ holds. So $\mathord{\underset{\smile}{\;}}$ is an independence relation. †

In the following sections, we will assume the hypotheses of the above theorem and use $\mathord{\underset{\smile}{\;}}$ as an independence relation. First, we discuss these hypotheses and provide some examples that satisfy them.

"amalgamation, joint embedding, and no maximal models"

These are a common set of assumptions when working with AECs that appear often in the literature; see [Sh394], [GV06a], and [GVV] for examples. Readers interested in work on AECs *without* these assumptions are encouraged to see [Sh576] or Shelah's work on good λ-frames in [Sh:h] and [JrSh875].

"fully $< \kappa$-tame" and "fully $< \kappa$-type short"

As discussed in [Bona].§3, these assumptions say that Galois types are equivalent to their small approximations. Without this equivalence, there is no reason to think that our nonforking, which is defined in terms of small approximations, would say anything useful about an AEC.

On the other hand, we argue that these properties will occur naturally in any setting with a notion of independence or stability theory. This is observed in the introduction to [GV06a]. Additionally, the following proposition says that the existence of a nonforking-like relation that satisfies stability-like assumptions implies tameness and some stability.

Proposition 5.3.2. *If there is a nonforking-like relation $\overset{*}{\underset{\smile}{\;}}$ that satisfies (U), (M), and $\kappa_\alpha(\overset{*}{\underset{\smile}{\;}}) < \infty$, then K is $(< \mu, \mu)$ tame for less than α length types for all regular $\mu \geq \kappa_\alpha(\overset{*}{\underset{\smile}{\;}})$.*

Proof: Let $p \neq q \in S^{<\alpha}(M)$ so their restriction to any smaller submodel is equal and let $\langle M_i \in K_{<\mu} : i < \mu \rangle$ be a resolution of M. By the local character, there are i_p and i_q such that p does not fork over M_{i_p} and q does not fork over M_{i_q}. By (M), both of the types don't fork over $M_{i_p+i_q}$ and, by assumption, $p \restriction M_{i_p+i_q} = q \restriction M_{i_p+i_q}$. Thus, by (U), we have $p = q$. †

The results of Chapter II allow us to get a similar result for type shortness.

The arguments of [MaSh285].4.14 and Theorem 6.3.1 show that this is enough to derive stability-like bounds on the number of Galois types.

"no weak κ order property"

In first order model theory, the order property and its relatives (the tree order property, etc) are well-studied

as the nonstructure side of dividing lines. In broader contexts such as ours, much less is known. Still, there are some results, such as Shelah [Sh:e].III, which shows that a strong order property, akin to getting any desired order of a certain size in an EM model, implies many models. Note that he does not explicitly work inside an AEC, but his proofs and definitions are sufficiently general and syntax free to apply here.

Ideally, the weak κ-order property could be shown to imply non-structure for an AEC. While this is not currently known in general, we have two special cases where many models follows by combinatorial arguments and the work of Shelah.

First, if we suppose that κ is inaccessible, then we can use Shelah's work to show that there are the maximum number of models in every size above κ. We will show that, given any linear order, there is an EM model with the order property for that order. This implies [Sh:e]'s notion of "weakly skeleton-like", which then implies many models by [Sh:e].III.24.

Proposition 5.3.3. *Let κ be inaccessible and suppose K has the weak κ-order property. Then, for all linear orders I, there is EM model M^*, small $N \prec M^*$, $p \neq q \in S(M)$, and $\langle a_i, b_i \in M^* : i \in I \rangle$ such that, for all $i, j \in I$,*

$$i \leq j \implies tp(a_i b_j / M) = p$$
$$i > j \implies tp(a_i b_j / M) = q$$

Thus, for all $\chi > \kappa$, K_χ has 2^χ nonisomorphic models.

We sketch the proof and refer the reader to [Sh:e] for more details.

Proof Outline: Let $p \neq q \in S(N)$ and $\langle a_i, b_i : i < \kappa \rangle$ witness the weak order property. Since K has no maximal models, we may assume that this occurs inside an EM model (see [Bonb] for details). In particular, there is some Φ proper for linear orders so $N \prec EM(\kappa, \Phi) \upharpoonright L$ that contains $\langle a_i, b_i : i < \kappa \rangle$, $L(\Phi)$ contains Skolem functions, and κ is indiscernible in $EM(\kappa, \Phi) \upharpoonright L$. Recall that, for $X \subset EM(\kappa, \Phi)$, we have $\text{Contents}(X) := \cap \{I \subset \kappa : X \subset |EM(I, \Phi)|\}$. By inaccessibility, we can thin out $\{\text{Contents}(a_i b_i) : i < \kappa\}$ to $\{\text{Contents}(a_i b_i) : i \in J\}$ that is a head-tail Δ system of size κ and are all generated by the same term and have the same quantifier free type in κ. Since κ is regular and $\text{Contents}(N)$ is of size $< \kappa$, we may further assume the non-root portion of this Δ system is above $\sup \text{Contents}(N)$.

By the definition of EM models, we can put in any linear order into $EM(\cdot, \Phi) \upharpoonright L$ and get a model in K. Thus, we can take the blocks that generate each $a_i b_i$ with $i \in J$ and arrange them in any order desired. In particular, we can arrange them such that they appear in the order given by I. Then, the order indiscernibility implies that the order property holds as desired.

We have shown the hypothesis of [Sh:e].III.24 and the final part of our hypothesis is that theorem's conclusion. †

We can also make use of these results without large cardinals. To do so, we 'forget' some of the tameness and type shortness our class has to get a slightly weaker relation. Suppose K is $< \kappa'$ tame and type short. Let λ be regular such that $\lambda^{\kappa'} = \lambda > \kappa'$. By the definitions, K is also $< \lambda$ tame and type short, so take λ to be our fixed cardinal κ. In this case, the ordered sequence constructed in the proof of Lemma 5.2.3 is actually of size $< \kappa'$. This situation allows us to repeat the above proof and construct 2^κ non-isomorphic models of size κ. Many other cardinal arithmetic set-ups suffice for many models.

(E)

This has already been discussed after Definition 2.3.1. Here we show that Existence, the simplicity style assumption that is equivalent to every models being κ saturated, follows from categoricity in a cardinal with favorable cardinal arithmetic.

Theorem 5.3.4. *Suppose K is an AEC satisfying the amalgamation property. If K is categorical in a cardinal λ satisfying $\lambda = \lambda^{<\kappa}$, then every member of $K_{\geq \chi}$ is κ-saturated, where $\chi = min\{\lambda, sup_{\mu<\kappa}(\beth_{(2^\mu)^+})\}$.*

Proof: First, note that by using the AP and the assumption $\lambda = \lambda^{<\kappa}$ we can construct a κ-saturated member of K_λ. Since this class is categorical, all members of K_λ are κ-saturated.

The easy case is when $\lambda < \chi$: Suppose $M \in K$ is not κ-saturated and $\|M\| > \lambda$. Then there is some small $M^- \prec M$ and $p \in S(M^-)$ such that p is not realized in M. Then let $N \prec M$ be any substructure of size λ containing M^-. Then N doesn't realize p, which contradicts its κ saturation.

For the hard part, suppose $M \in K$ is not κ saturated and $\|M\| \geq \sup_{\mu<\kappa}(\beth_{(2^\mu)^+})$. There is some small $M^- \prec M$ and $p \in S(M^-)$ such that p is not realized in M. We define a new class $(K^+, \prec^+)$ that depends on K, p and M^- as follows:

$L(K^+) := L(K) \cup \{c_m : m \in |M^-|\}$ by

$$
\begin{aligned}
K^+ \quad = \quad &\{N : N \text{ is an } L(K^+) \text{ structure st } N \restriction L(K) \in K, \text{ there exists} \\
&h : M^- \to N \restriction L(K) \text{ a } K\text{-embedding such that } h(m) = (c_m)^N \\
&\text{for all } m \in M^- \text{ and } N \restriction L(K) \text{ omits } h(p)\}. \\
N_1 \prec^+ N_2 \quad \Longleftrightarrow \quad &N_1 \restriction L(K) \prec N_2 \restriction L(K) \text{ and } N_1 \subset_{L(K^+)} N_2.
\end{aligned}
$$

This is clearly an AEC with $LS(K^+) = \|M^-\| + LS(K) < \kappa$ and $\langle M, m\rangle_{m\in|M^-|} \in K^+$.

By Shelah's presentation Theorem K^+ is a $PC_{\mu,2^\mu}$ for $\mu := LS(K^+)$. By Thoerems VII.5.5(2) and VII.5.5(6) of [Sh:c] the Hanf number of K^+ is $\leq \beth_{(2^\mu)^+} \leq \chi$.

Thus, K^+ has arbitrarily large models. In particular, there exists $N^+ \in K_\lambda^+$. Then $N^+ \restriction L(K) \in K_\lambda$ is not κ-saturated as it omits its copy of p. $\qquad\qquad\qquad\qquad\dagger$

Remark 5.3.5. *While for the rest of the results we assume that K satisfies Hypotheses 5.1.2, in the proof of Theorem 5.3.4 we use only the amalgamation property and also avoid any use of tameness or type shortness.*

Before continuing, we also identify a few contexts which are known to satisfy this hypothesis, especially (1), (2), and (3) of Theorem 5.3.1.

- **First order theories** Since types are syntactic and over sets, they are $< \aleph_0$ tame and $< \aleph_0$ type short and (4) follows by compactness. Additionally, when (3) holds, the theory is stable so coheirs are equivalent to non-forking. While we don't claim to have discovered anything new about first-order theories, formally speaking our framework apply to K_T where T is a superstable first-order theory and K_T is the class of $|T|^+$-saturated models (our κ is $|T|^+$).

- **Large cardinals** Chapter IV of this thesis proves that (1),(2), and Extension hold for any AEC K that are essentially below a strongly compact cardinal κ (this holds, for instance, if $LS(K) < \kappa$). Slightly weaker (but still useful) versions of (1) and (2) also hold if κ is measurable or weakly compact. See Section 5.6 for more.

- **Homogeneous model theory** The homogeneity of the monster model ensures that the types are tame and type short. Hyttinen and Shelah [HySh629]

- **Zilber's pseudoexponentiation** See page 190 in Baldwin's book [Bal09].

5.4 Getting Local Character

Local character is a very important property for identifying dividing lines. In the first order context, some of the main classes of theories–superstable, strictly stable, strictly simple, and unsimple–can be identified by the value of $\kappa(T)$. By finding values for $\kappa_\alpha(\bot)$ under different hypotheses, we get candidates for dividing lines in AECs.

Readers familiar with first order stability theory will recall that there is a notion of an heir of a type that is the dual notion to coheir, which our nonforking is based on. Heir is equivalent to to the notion of coheir under the assumption of no order; see [Pil83].1 and .2 as a reference. We develop an AEC version of heir and explore its relation with nonforking. We further show that there is an order property that implies their equivalence. This equivalence allows us to adapt an argument of [ShVi635] to calculate $\kappa(\bot)$ from categoricity. In this discussion, we only assume the properties of nonforking that follow immediately from the definition, like those in Theorem 5.2.4, and explicitly state any other assumptions. In particular, note that Theorem 5.4.6 doesn't assume (E), the failure of the weak κ-order property, or tameness or type shortness.

Recall that 'small' refers to objects of size $< \kappa$.

Definition 5.4.1. *We say that $p \in S^I(N)$ is an* heir *over $M \prec N$ iff for all small $I_0 \subset I$, $M^- \prec M$, and $M^- \prec N^- \prec N$ (with M^- possibly being empty), there is some $f : N^- \to_{M^-} M$ such that $f(p^{I_0} \upharpoonright N^-) \leq p$; that is, $f(p)^{I_0} \upharpoonright f(N^-) = p^{I_0} \upharpoonright f(N^-)$. We also refer to this by saying p is a heir of $p \upharpoonright M$.*

$$
\begin{array}{ccc}
M & \longrightarrow & N \\
\uparrow \;\; \nwarrow {}^{f} & & \uparrow \\
M^- & \longrightarrow & N^-
\end{array}
$$

At first glance, this property seems very different from the first order version of heir. However, if we follow the remark after Theorem 5.3.1, we can think of restrictions of p as formulas and small models as parameters. Then, M^- is a parameter from M, N^- is a parameter from N, $f(N^-)$ is the parameter from M that corresponds to N^- (notice that it fixes M^-), and $f(p \upharpoonright N^-) \leq p$ witnesses that it the original formula $p \upharpoonright N^-$ is still in p with a parameter from M.

If we restrict ourselves to models, then the notions of heir over and nonforking (coheir over) are dual with no additional assumptions.

Proposition 5.4.2. *Suppose $M_0 \prec M, N$. Then $tp(M/N)$ does not fork over M_0 iff $tp(N/M)$ is an heir over M_0.*

Proof: First, suppose that $M \underset{M_0}{\bot} N$ and let $a \in |N|$ be of length $< \kappa$. Let $M_0^- \prec M_0$ and $M^- \prec M$ both be of size $< \kappa$ such that $M_0^- \prec M^-$. Find $N^- \prec N$ of size $< \kappa$ containing M_0^- and a. By the definition of nonforking, $tp(M^-/N^-)$ is realized in M_0. This means that there is $g \in Aut_{M^-}\mathfrak{C}$ such that

$g(M^-) \prec M_0$. Set $f = g \restriction M^-$. Then $f : M^- \to_{M_0^-} M_0$ such that $f(tp(a/M^-)) = tp(a/f(M^-))$. Since a, M_0^-, and M^- were arbitrary, $tp(N/M)$ is an heir over M_0.

Second, suppose that $tp(N/M)$ is an heir over M_0. Let $b \in M$ and $N^- \prec N$ both be of size $< \kappa$. Since M is a model, we may expand b to a model $M^- \prec M$ of size $< \kappa$. Then, if we can realize $tp(M^-/N^-)$ in M_0, we can find a realization of $tp(b/N^-)$ there as well. By assumption, there is some $f : M^- \to M_0$ such that $tp(f(N^-)/f(M^-)) = tp(N^-/f(M^-))$. This type equality means that there is some $g \in Aut_{f(M^-)}\mathfrak{C}$ such that $g(f(N^-)) = N^-$. Thus, $g \circ f$ is in $Aut_{N^-}\mathfrak{C}$ and sends M^- to $f(M^-) \prec M_0$. Thus, $tp(M^-/N^-) = tp(f(M^-)/N^-)$ and is realized in M_0, as desired. †

This proposition was proven just from the definitions, without assuming any tameness or type shortness. If we assume even the weak symmetry (S^*), then we have that nonforking and heiring are equivalent for models. Assuming full symmetry (S) is enough to get the full implication in one direction.

Theorem 5.4.3. *Suppose $\perp$ satisfies (S). If $p \in S(N)$ and $M \prec N$, then p does not fork over M holds implies p is an heir over M.*

Proof: Suppose $p \in S(N)$ does not fork over M. Then, given A that realizes p, we have $A \underset{M}{\perp} N$. By (S), we can find $M^+ \succ M$ containing A such $N \underset{M}{\perp} M^+$. By Proposition 5.4.2, we then have $tp(M^+/N)$ is an heir over M. By motonicity, $p = tp(A/N)$ is an heir over M. †

However, for the other direction, this does not suffice. It would be possible to completely redevelop the stability theory of the previous sections for the notion of heiring, but this would not help us understand the real connection between nonforking and heiring. Instead, we draw a parallel to the first order case. There, the equivalence of heir and coheir uses the order property, as does the first order version of Lemma 5.2.3 above. Following this, we introduce a new order property, order$_2$, that characterizes the relationship between nonforking and heiring. We refer to order$_2$ as "an order property" because, like Definition 5.2.2, it is witnessed by a sequence whose order is semantically definable inside of the AEC.

Definition 5.4.4. *We say that an AEC K has the (λ, α)-order$_2$ property iff there are parameters b and $\langle b_i : i < \alpha \rangle$ and models $\langle N_i \in K : i < \alpha \rangle$ such that $\ell(b_i) + \|N_i\| < \lambda$ and, for all $i, j < \alpha$, we have*

$$i \leq j \text{ iff } b_j \models tp(b/N_i)$$

We now prove that no order$_2$ property means that heiring implies nonforking. This follows the first order version as presented in [Pil83].2.2.

Theorem 5.4.5. *Let K be an AEC and $M \prec N$ be models such that M is κ saturated. If there is $p \in S(N)$ that is a heir over M and also forks over M, then K has the (κ, κ)-order$_2$ property, and it is witnessed in M.*

Proof: Suppose that $b \models p$. Since $\neg(b \underset{M}{\perp} N)$, there is some $N^- \prec N$ such that $tp(b/N^-)$ is not realized in M. We are going to construct two sequences $\langle b_i \in |M| : i < \kappa \rangle$ and increasing $\langle N_i^- \prec M : i < \kappa \rangle$ that witness the (κ, κ)-order$_2$ property.

Suppose that we have our sequences defined for all $j < i$ for some fixed $i < \kappa$. Set small $M_i^+ \prec M$ to contain all $\{N_j^-, b_j : j < i\}$ and $N_i^+ \prec N$ to contain M_i^+ and N^-, both of size $< \kappa$; if $i = 0$, then we

just take $M_i^+ = \emptyset$ and $N_i^+ = N^-$. Since $tp(b/N)$ is a heir over M, we can find some $f_i : N_i^+ \to_{M_i^+} M$ such that $tp(f_i(b)/f_i(N_i^+)) = tp(b/f_i(N_i^+))$. Set $N_i^- = f_i(N_i^+)$ and extend f_i to an automorphism f_i^+ of $\mathfrak{C}$. By the κ saturation of M, there is $b_i \in |M|$ that realizes $tp(b/N_i^-)$.

Now we want to show that these exhibit the order$_2$ property:

i $\leq$ j: By construction, $N_i^- \prec N_j^-$, so, in particular, $tp(b/N_i^-) \leq tp(b/N_j^-)$. Also, $b_j \vDash tp(b/N_j^-)$, so we have $b_j \vDash tp(b/N_i^-)$.

i $>$ j: Suppose $b_j \vDash tp(b/N_i^-)$. This means

$$
\begin{aligned}
b_j &\vDash tp(b/N_i^-) \\
b_j &\vDash tp(f_i^+(b)/N_i^-) && \text{by the definition of heir} \\
(f_i^+)^{-1}(b_j) &\vDash tp(b/N_i^+) \\
b_j &\vDash tp(b/N_i^+) && b_j \in |M_i^+| \\
b_j &\vDash tp(b/N^-) && N^- \prec N_i^+
\end{aligned}
$$

which contradicts our assumption that $tp(b/N^-)$ is not realized in M.

So $\langle b_i, N_i^- : i < \kappa \rangle$ witnesses the (κ, κ)-order$_2$ property. †

Now that we have established an equivalence between nonforking and being an heir, we aim to derive local character. For this, we use heavily the proof [ShVi635].2.2.1, which shows that, under certain assumptions, the universal local character cardinal for non-splitting is ω. Examining the proof, much of the work is done by basic independence properties–namely (I), (M), and (T)–and the other assumptions on K–namely categoricity, amalgamation, and EM models, which follow from no maximal models. Only in case (c), defined below, do they need the exact definition of their independence relation (non μ-splitting) and GCH. In this case, we can use the definition of heir to complete the proof.

Theorem 5.4.6. *Suppose that K has no (κ, κ)-order$_2$ property, is categorica lin some $\lambda \geq \kappa$, and is stable in κ. Then $\kappa_\omega^*(\bot) = \omega$. That is, if*

1. *$\langle M_i \in K_\mu : i \leq \alpha \rangle$ is increasing and continuous;*

2. *each M_{i+1} is universal over M_i and κ saturated;*

3. *$cf\, \alpha = \alpha < \mu^+ \leq \lambda$; and*

4. *$p \in S^{<\omega}(M_\alpha)$*

then, for some $i < \alpha$, p does not fork over M_i

Proof: Deny and set $M = M_\alpha$. As in [ShVi635], we consider the three following cases:

(a) for all $i < \alpha$, $p \restriction M_i$ does not fork over M_0;

(b) (a) is impossible and for all $i < \alpha$, $p \restriction M_{2i+1}$ forks over M_{2i} and M_{2i+2} does not fork over M_{2i+1}

(c) (a) and (b) are impossible and $\alpha = \mu \geq \kappa$ and for all $i < \alpha$, $p \restriction M_{i+1}$ forks over M_i

Shelah and Villaveces first show that, using only (M), (I), and (T), one of these three cases must hold. Then, cases (a) and (b) are eliminated using categoricity and EM models, both of which are part of the assumptions. Thus, we can assume that we are in case (c).

Then, by Theorem 5.4.5 and the assumption of no (κ, κ)-order$_2$ property, we know that $p \restriction M_{i+1}$ is not a heir over M_i for all $i < \alpha$. Find the minimum σ such that $2^\sigma > \kappa$. Then $\sigma \leq \kappa$ and $2^{<\sigma} \leq \kappa$. We are going to contradict stability in κ by finding 2^σ many types over a model of size $2^{<\sigma}$.

Step 1: We define $\langle M^i \prec N^i \mid i < \alpha \rangle$ as follows: for each $i < \alpha$, since $p \restriction M_{i+1}$ is not a heir over M_i, there exists some $M^i \prec N^i \in K_{<\kappa}$ such that $M^i \prec M_i$ and $N^i \prec M_{i+1}$ and for any $h : N^i \to_{M^i} M_i$, we have $h(p \restriction N^i) \neq p \restriction h(N^i)$.

Now define $\langle \hat{M}_i \prec \hat{N}_i \in K_{<\kappa} \mid i < \alpha \rangle$ increasing and continuous and $g_i : \hat{N}_i \to_{\hat{M}_i} \hat{M}_{i+1}$ by setting $\hat{M}_0 = M^0$ and $\hat{N}_0 = N^0$ and taking unions at limits. If we have $\hat{M}_i$ and $\hat{N}_i$ defined, then we can use the saturation of $M_{i+1} \succ \hat{M}_i$ to find some $g_i : \hat{N}_i \to_{\hat{M}_i} M_{i+1}$. Then pick $\hat{M}_{i+1} \prec M_{i+1}$ to contain $g_i(\hat{N}_i)$ and M^{i+1} and $\hat{N}_{i+1} \prec M_{i+2}$ to contain N^{i+1} and $\hat{M}_{i+1}$.

Now that we have finished this construction, notice that $g_i \restriction N^i : N^i \to_{M^i} \hat{M}_{i+1} \prec M_{i+1}$, so $g_i(p \restriction N^i) \neq p \restriction g_i(N^i)$. Since inequality of types always transfers up, we have $g_i(p \restriction \hat{N}_i) \neq p \restriction g_i(\hat{N}_i)$.

Step 2: First, we relabel elements as standard. We change:

$$
\begin{array}{ccccccc}
\ldots \longrightarrow & \hat{N}_{2i} & \longrightarrow & \hat{N}_{2i+1} & \longrightarrow & \hat{N}_{2i+2} & \longrightarrow \ldots \\
& \uparrow & {\scriptstyle g_{2i}} & \uparrow & {\scriptstyle g_{2i+1}} & \uparrow & \\
\ldots \longrightarrow & \hat{M}_{2i} & \longrightarrow & \hat{M}_{2i+1} & \longrightarrow & \hat{M}_{2i+2} & \longrightarrow \ldots
\end{array}
$$

to

$$
\begin{array}{ccccccc}
\ldots \longrightarrow & N_{2i} & \longrightarrow & N_{2i+1} & \xrightarrow{f_{2i+1}} & N_{2i+2} & \longrightarrow \ldots \\
& \uparrow & {\scriptstyle g_{2i}} & \uparrow & & \uparrow & \\
\ldots \longrightarrow & M_{2i} & \longrightarrow & M_{2i+1} & \longrightarrow & M_{2i+2} & \longrightarrow \ldots
\end{array}
$$

by setting $f_{2i+1} = g_{2i+1}^{-1}$; $M_{2i+2} = g_{2i+1}^{-1}(\hat{M}_{2i+2})$, $M_{2i+1} = \hat{M}_{2i+1}$; $N_{2i+2} = g_{2i+1}^{-1}(\hat{N}_{2i+2})$, and $N_{2i+1} = \hat{N}_{2i+1}$. We do this so we have identities where we want them and embeddings (the f_{2k+1}'s) on paths we ignore. Now define, for each $i < \alpha$, $N_i \prec N_i^1, N_i^2 \prec N_{i+1}$ all in $K_{<\kappa}$ with $\langle N_i \mid i < \alpha \rangle$ increasing and continuous and $h_i : N_i^1 \to_{N_i} N_i^2$ such that $h_i(p \restriction N_i^1) \neq p \restriction h_i(N_i^1)$. This is done by setting $N_i = \hat{M}_{2i}$, $N_i^1 = \hat{N}_{2i}$, $N_i^2 = \hat{M}_{2i+1}$, and $h_i = g_{2i}$.

Step 3: We now construct a tree of types of height κ that will all be different at the top. For each $\eta \in {}^{\leq \sigma}2$, we are going to construct

- $\widehat{h}_\eta \in Aut\,\mathfrak{C}$;

- increasing, continuous $N_\eta \in K_{<\kappa}$ such that $\widehat{h}_\eta(N_{\ell(\eta)}) = N_\eta$; and

- increasing $p_\eta \in S(N_\eta)$.

We work by induction on the length of η.

- When $\eta = 0$, set $\widehat{h}_\emptyset = \mathrm{id}_\mathfrak{C}$, $N_\emptyset = N_0$, and $p_\emptyset = p \restriction M_0$.

- When η is the successor of ν, set $\widehat{h}_{\nu^\frown 0} = \widehat{h}_\nu$ and $\widehat{h}_{\nu^\frown 1} = \widehat{h}_\nu \circ h_{\ell(\nu)}$. Then set $N_{\nu^\frown i} = \widehat{h}_{\nu^\frown i}(N_{\ell(\nu+1)})$, as required, and $p_\eta = \widehat{h}_\eta(p \upharpoonright N_{\ell}(\eta))$.

- When η is a limit, set $N_\eta = \cup_{\alpha < \ell(\eta)} N_{\eta\upharpoonright\alpha}$. Then we have that $\langle \widehat{h}_{\eta\upharpoonright\alpha} \upharpoonright N_{\eta\upharpoonright\alpha} : \alpha < \ell(\eta) \rangle$ is an increasing sequence, so set $\widehat{h}_\eta$ to be any automorphism of $\mathfrak{C}$ extending their union and $p_\eta = \widehat{h}_\eta(p \upharpoonright N_{\ell(\eta)})$.

Once we have completed this construction, note that our choice of σ guarantees that there is some M^* of size κ such that $N_\eta \prec M^*$ for all $\eta \in {}^{<\sigma}2$. Thus, $N_\eta \prec M^*$ for all $\eta \in {}^\sigma 2$. Then, for each $\eta \in {}^\sigma 2$, we can extend p_η to some $p_\eta^* \in S(M^*)$. Once we prove the following claim, we will have contradicted stability in κ since $2^\sigma > \kappa$.

Claim: If $\eta \neq \eta' \in {}^\sigma 2$, then $p_\eta^* \neq p_{\eta'}^*$.

Set $\nu = \eta \cap \eta'$ and $i = \ell(\nu)$. WLOG, $\nu^\frown 0 \subset \eta$ and $\nu^\frown 1 \subset \eta'$. From their construction, we know the following things about these types:

- $p_{\nu^\frown 0} = \widehat{h}_{\nu^\frown 0}(p \upharpoonright N_{i+1}) = \widehat{h}_\nu(p \upharpoonright N_{i+1}) \leq p_\eta$;

- $p_{\nu^\frown 1} = \widehat{h}_{\nu^\frown 1}(p \upharpoonright N_{i+1}) = \widehat{h}_\nu(h_i(p \upharpoonright N_{i+1})) \leq p_{\eta'}$;

- $p \upharpoonright N_i^2 \leq p \upharpoonright N_{i+1}$;

- $h_i(p) \upharpoonright N_i^2 \leq h_i(p) \upharpoonright N_{i+1}$; and

- $p \upharpoonright N_i^2 \neq h_i(p) \upharpoonright N_i^2$.

Since inequality of types transfers upwards, this is enough. The bottom three lines imply that $p \upharpoonright N_{i+1} \neq h_i(p) \upharpoonright N_{i+1}$. Since the first two lines show that the same map $\widehat{h}_\nu$ maps the lefthand-side as a subtype of p_η and the righthand-side as a suptype of $p_{\eta'}$, this finishes the claim and the proof. †

This construction could not go further than κ many steps because the definition of heir requires all of the models and tuples involved to be of size $< \kappa$. Thus, we need to know that stability fails at κ. If we knew that nonforking and nonsplitting were the same, instead of just nonforking and heiring, then we would have a more general argument. The connection between these two notions of independence and other is explored more in [BGKV].

Once we have the universal local character, we can get results on the uniqueness of limit models. While many results in this area are described in Chapter II, the most relevant for our context is the proof that is outlined in [Sh:h].II.4 and detailed in the next chapter. There, Shelah's frames are used to create a matrix of models to show that limit models are isomorphic. Inspecting the proof, the only property used that is not a part of an independence property is a stronger continuity restricted to universal chains. This follows from universal local character.

Corollary 5.4.7. *Suppose there is some $\kappa > LS(K)$ such that*

1. *K is fully $< \kappa$-tame;*

2. *K is fully $< \kappa$-type short;*

3. *K doesn't have the weak κ-order property or the (κ, κ)-order$_2$ property;*

4. $\perp$ *satisfies* (E); *and*

5. *it is categorical in some* $\lambda > \kappa$

Then K *has a unique limit model in each size in* $[\kappa, \lambda)$. *Moreover, if* λ *is a successor, then* K *has unique limit models in each size above* κ.

Proof: The first part follows from Theorems 5.3.1, 5.4.5, and 5.4.6 and Lemma 6.8.1. The moreover follows from the categoricity transfer of [GV06a]. †

Note that the uniqueness of limit models as stated does not follow trivially from categoricity because it requires that the isomorphism fixes the base.

5.5 The U-Rank

Independence relations and ranks go hand in hand in first order theories: in the appropriate contexts, splitting is equivalent to an increase of the two-rank [Gro1X].6.4.4, non-weak minimality to an increase of the Deg [Sh31].4.2, forking to an increase in the local rank [Sh:c].Theorem III.4.1.

Here we develop a U-rank for our forking and show that, under suitable conditions, it behaves as desired. The U-rank was first introduced by Lascar [Las75] for first order theories and first applied to AECs by [Sh394]. They have also been studied by Hyttinen, Kesala, and Lessman in various nonelementary contexts; see [Les00], [Les03], [HyLe02], and [HyKe06].

For this section, we add the hypotheses of the main theorem so that $\perp$ will be an independence relation. Indeed, the results of this section do not use our specific definition of nonforking, but just that it satisfies the axioms of an independence relation given in Definition 2.3.1.

Hypothesis 5.5.1. *Suppose that there is some* $\kappa > LS(K)$ *such that*

1. K *is fully* $< \kappa$-*tame;*

2. K *is fully* $< \kappa$-*type short;*

3. K *doesn't have the weak* κ-*order property; and*

4. $\perp$ *satisfies* (E).

Definition 5.5.2. *We define* U *with domain a type and range an ordinal or* ∞ *by, for any* $p \in S(M)$

1. $U(p) \geq 0$;

2. $U(p) \geq \alpha$ *limit iff* $U(p) \geq \beta$ *for all* $\beta < \alpha$;

3. $U(p) \geq \beta + 1$ *iff there is* $M' \succ M$ *with* $\|M'\| = \|M\|$ *and* $p' \in S(M')$ *such that* p' *is a forking extension of* p *and* $U(p') \geq \beta$;

4. $U(p) = \alpha$ *iff* $U(p) \geq \alpha$ *and* $\neg(U(p) \geq \alpha + 1)$; *and*

5. $U(p) = \infty$ *iff* $U(p) \geq \alpha$ *for every* α.

First we prove a few standard rank properties. The first several results are true without the clause about the sizes of the model, but this is necessary later when we give a condition for the finiteness of the rank for Lemma 5.5.8.

Lemma 5.5.3 (Monotonicity). *If $M \prec N$, $p \in S(M)$, $q \in S(N)$, and $p \leq q$, then $U(q) \leq U(p)$.*

Proof: We prove by induction on α that $p \leq q$ implies that $U(q) \geq \alpha$ implies $U(p) \geq \alpha$. For limit α, this is clear, so assume $\alpha = \beta + 1$ and $U(q) \geq \beta + 1$. Then there is a $N' \succ N$ and $q^+ \in S(N')$ that is a forking extension of q and $IU(q^+) \geq \beta$. By (M), it is also a forking extension of p. Then $U(p) \geq \alpha$ as desired. †

Lemma 5.5.4 (Invariance). *If $f \in Aut\mathfrak{C}$ and $p \in S(M)$, then $U(p) = U(f(p))$.*

Proof: Clear. †

Proposition 5.5.5 (Ultrametric). *The U rank satisfies the ultrametric property; that is, if we have $M \prec N_i$, $p \in S(M)$ and distinct $\langle q_i \in S(N_i) \mid i < \alpha \rangle$ are such that $a \models p$ iff there is an $i_0 < \alpha$ such that $a \models q_{i_0}$, then we have $U(p) = \max_{i < \alpha} U(q_i)$.*

Note that, as always, we assume α is well below the size of the monster model.

Proof: We know that $p \leq q_i$ for all $i < \alpha$, so, by Lemma 5.5.3, we have $\max_{i < \alpha} U(q_i) \leq U(p)$. Since we have a monster model, we can find some $N^* \in K$ that contains all N_i. By (E), we can find some $p^+ \in S(N^*)$ such that p^+ is a non-forking extension of p. Now, let $a \models p^+$. Since $p \leq p^+$, $a \models p$. Since '$p(\mathfrak{C}) = \cup_{i < \alpha} q_i(\mathfrak{C})$,' there is some $i_0 < \alpha$ such that $a \models q_{i_0}$. But then $a \underset{M}{\downarrow} N^+$ implies $a \underset{M}{\downarrow} N_{i_0}$ by (M), so $tp(a/N_{i_0}) = q_{i_0}$ does not fork over M. Then

$$U(p) = U(q_{i_0}) = \max_{i < \alpha} U(q_i)$$

†

We want to show that same rank extensions correspond exactly to non-forking when the U-rank is ordinal valued. One direction is clear from the definition. For the other, we generalize first order proofs to the AEC context; this proof follows the one in [Pil83]. First, we prove the following lemma.

Lemma 5.5.6. *Let $N_0 \prec N_1 \prec \bar{N}_1$, $N_0 \prec \bar{N}_0 \prec \bar{N}_1$, and $N_0 \prec N_2$ be models with some $c \in \bar{N}_0$. If*

$$N_1 \underset{N_0}{\downarrow} \bar{N} \text{ and } N_2 \underset{\bar{N}_0}{\downarrow} \bar{N}_1$$

then there is some N_3 extending N_1 and N_2 such that

$$c \underset{N_2}{\downarrow} N_3$$

Proof: We can use (S) twice on $N_2 \underset{\bar{N}_0}{\downarrow} \bar{N}_1$ to find $\bar{N}_2$ extending N_2 and $\bar{N}$ such that $\bar{N}_2 \underset{\bar{N}}{\downarrow} \bar{N}_1$. This contains c, so (M) implies that $N_2 c \underset{\bar{N}}{\downarrow} \bar{N}_1$. By applying (S) to the other nonforking from our hypothesis, we know $\bar{N} \underset{N_0}{\downarrow} N_1$. By (T), this means that $N_2 c \underset{N_0}{\downarrow} N_1$.

Applying (S) to this, there is some N_3' extending N_2 and containing c such that $N_1 \underset{N_0}{\downarrow} N_3'$. By (M), we have that $N_1 \underset{N_2}{\downarrow} N_3'$. Applying (S) one final time, we can find an N_3 extending N_1 and N_2 such that $c \underset{N_2}{\downarrow} N_3$. †

Theorem 5.5.7. *Let $p \in S(M_0)$ and $q \in S(M_1)$ such that $p \le q$ and $U(p), U(q) < \infty$. Then*

$$U(p) = U(q) \iff q \text{ is a nonforking extension of } p$$

Proof: By definition, $U(p) = U(q)$ implies q does not fork over M_0. For the other direction, we show by induction on α that, for any q that is a nonforking extension of some p, $U(p) \ge \alpha$ implies $U(q) \ge \alpha$.

If α is 0 or limit, this is straight from the definition.

Suppose that $U(p) \ge \alpha + 1$. Then, there are $M_2 \succ M_0$ and $p_1 \in S(M_2)$ such that p_1 is a forking extension of p and $U(p_1) \ge \alpha$.

Claim: We may pick M_2 and p_1 such that there is a M_3 extending M_1 and M_2 and $q_1 \in S(M_3)$ so

- $q_1 \ge q, p_1$; and

- q_1 does not fork over M_2.

Once we prove this claim, we will be done.

Assume for contradiction that q_1 does not fork over M_1. By [BGKV].6.9, a right version of transitivity also holds of our nonforking:

$$\text{if } A \underset{M_0}{\downarrow} M_1 \text{ and } A \underset{M_1}{\downarrow} M_2 \text{ with } M_0 \prec M_1 \prec M_2, \text{ then } A \underset{M_0}{\downarrow} M_2$$

Thus, q_1 would also not fork over M_1. By (M), this would imply that p_1 does not fork over M_0, a contradiction. Thus, q_1 is a forking extension of q of U rank at least α. Thus, $U(q) \ge \alpha + 1$.

To prove the claim, let d realize q and d' realize p_1. Since both of these types extend p, there is some $f \in Aut_{M_0}\mathfrak{C}$ such that $f(d') = d$. Set $M_2' = f(M_2)$. We know that $d \underset{M_0}{\downarrow} M_1$, so by (S), there is some $\bar{M}_0 \succ M_0$ that contains d so $M_1 \underset{M_0}{\downarrow} \bar{M}_0$. Pick $\bar{M}_1 \prec \mathfrak{C}$ that contains $\bar{M}_0$ and M_1. By (E), there is some M_2'' so that $tp(M_2'/\bar{M}_0) = tp(M_2''/\bar{M}_0)$ and $M_2'' \underset{\bar{M}_0}{\downarrow} \bar{M}_1$. Let $g \in Aut_{\bar{M}_0}\mathfrak{C}$ such that $g(M_2') = M_2''$; note that this fixes d.

We may now apply our lemma. This means there is some M_3 that extends M_2'' and M_1 such that $d \underset{M_2''}{\downarrow} M_3$. Now this proves our claim with M_2'' and $tp(d/M_2'') = g(f(p_1))$ and witnesses M_3 and $q_1 = tp(d/M_3)$. †

We now give a condition for the U rank to be ordinal valued, as in [Sh394].5. First, note that clause about the model sizes in the definition of U gives a bound for the rank.

Lemma 5.5.8 (Ordinal Bound). *If $M \in K_\mu$ and $p \in S(M)$, then $U(p) > (2^\mu)^+$ implies $U(p) = \infty$.*

Theorem 5.5.9 (Superstability). *Let $M \in K_\mu$ and $p \in S(M)$. Then the following are equivalent:*

1. $U(p) = \infty$.

2. *There is an increasing sequence of types $\langle p_n : n < \omega \rangle$ such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for all $n < \omega$.*

Proof: First, suppose $U(p) = \infty$ and set p_0. We will construct our sequence by induction such that $U(p_n) = \infty$. Then $U(p_n) > (2^\mu)^+ + 1$, so there is a forking extension p_{n+1} with the same sized domain and $U(p_{n+1}) > (2^\mu)^+$. But then $U(p_{n+1}) = \infty$ and out induction can continue.

Second, suppose we have such a sequence $\langle p_n : n < \omega \rangle$ and we will show, by induction on α, the $U(p_n) \geq \alpha$ for all $n < \omega$. The 0 and limit stages are clear. At stage $\alpha + 1$, p_{n+1} is a forking extension of p_n with rank at least α. Thus, $U(p_n) \geq \alpha + 1$. †

Ranks in a tame AEC have also been explored by Lieberman [Lie13]. Under a tameness assumption, he introduces a series of ranks that emulate Morley Rank.

Definition 5.5.10 ([Lie13].3.1)*. Let $\lambda \geq \kappa$, where K is κ-tame. For $M \in K_\lambda$ and $p \in S(M)$, we define $R^\lambda(p)$ inductively by*

- $R^\lambda[p] \geq 0$;

- $R^\lambda[p] \geq \alpha$ *for limit α iff $R^\lambda[p] \geq \beta$ for all $\beta < \alpha$; and*

- $R^\lambda[p] \geq \beta + 1$ *iff there is $M' \succ M$ and $\langle p_i \in S(M') : i < \lambda^+ \rangle$ such that $p \leq p_i$ and $R^\lambda[p_i] \geq \beta$ for all $i < \lambda^+$.*

If $\|M\| > \lambda$ and $p \in S(M)$, then

$$R^\lambda[p] = \min\{R^\lambda[p \upharpoonright N] : N \prec M, \|N\| = \lambda\}$$

Our U-rank dominates these Morley Ranks at least for domains of size λ. Thus, the finiteness of the U-rank, which follows from local character, implies us that an AEC is totally transcendental and that the stability transfer results of [Lie13].§5 apply.

Theorem 5.5.11. *Let $M \in K_\lambda$, $p \in S(M)$, and $\lambda \geq \kappa$. Then $U(p) \geq RM^\lambda(p)$.*

Proof: We prove, simultaneously for all types, that $RM^\lambda(p) \geq \alpha$ implies $U(p) \geq \alpha$ for all α by induction. For $\alpha = 0$ or limit, this is easy.

Suppose $RM^\lambda(p) \geq \alpha + 1$. Let M' and $\langle p_i \in S(M') : i < \lambda^+ \rangle$ witness this. p has a unique nonforking extension to M', call it p^*. Thus, almost all of the p_i fork over M; let p_{i_0} be one of them. Then, $p_{i_0} \neq p^*$, so there is some $M_0 \prec M'$ of size $< \kappa$ such that $p_{i_0} \upharpoonright M_0 \neq p^* \upharpoonright M_0$. Let $M'' \prec M'$ contain M and M_0 such that $\|M\| = \|M''\|$ and $p' = p_{i_0} \upharpoonright M''$. Then

- p' extends p;

- p' is a forking extension of p because it differs from the nonforking extension, $p^* \upharpoonright M_{i_0}$; and

- $RM^\lambda(p') \geq RM^\lambda(p_{i_0})$ by [Lie13].3.3. So $RM^\lambda(p') \geq \alpha$. By induction, this means $U(p') \geq \alpha$.

So $U(p) \geq \alpha + 1$, as desired. †

5.6 Large cardinals revisited

In this section, we discuss the behavior of non-forking in the presence of large cardinals. We return to just assuming Hypothesis 5.1.2, that K satisfies amalgamation, joint embedding, and no maximal models.

In Chapter IV, we showed that the tameness and types shortness part of the hypothesis of Theorem 5.3.1 follows from κ being strongly compact. We now explore how this and other large cardinals affect the other parts of that hypothesis.

The following details a construction that will be used often in the following proof. This construction and the proof of the following theorem draw inspiration from [MaSh285]. Suppose that $M \prec N$ and U is a κ complete ultrafilter over I. Then Łoś' Theorem for AECs states that the canonical ultrapower embedding $h : N \to \Pi N/U$ that takes n to the constant function $[i \mapsto n]_U$ is a K-embedding. We can expand h to some h^+ that is an $L(K)$ isomorphism with range $\Pi N/U$ and set $N^U := (h^+)^{-1}[\Pi N/U]$. This is a copy of the ultraproduct that actually contains N. Similarly, we can set $M^U := (h^+)^{-1}[\Pi N/U]$. The following claim is key.

Claim: $M^U \underset{M}{\downarrow} N$.

Proof: Let small $N^- \prec N$ and $a \in M^U$. Then $h^+(a) = [f]_U$ for some $[f]_U \in \Pi M/U$. Denote $tp(a/N^-)$ by p. Then, by Łoś' Theorem, version 2, we have

$$
\begin{aligned}
a &\vDash p \\
h^+(a) = [f]_U &\vDash h^+(p) = h(p) \\
X &:= \{i \in I : f(i) \vDash p\} \in U
\end{aligned}
$$

Since $[f]_U \in \Pi M/U$, there is some $i_0 \in X$ such that $f(i_0) \in M$. Then $f(i_0) \vDash p$ as desired. †

We now show that non-forking is very well behaved in the presence of a strongly compact cardinal. Note that the second part says that the local character property holds very strongly if the type does not fork over its domain and the third part improves on Theorem 5.3.4 by showing that categoricity implies an analogue of superstability instead of just an analogue of simplicity.

Theorem 5.6.1. *Suppose κ is strongly compact and K is an AEC such that $LS(K) < \kappa$. Then*

1. *$\downarrow$ satisfies Extension.*

2. *If $M = \cup_{i<\alpha} M_i$, $p \in S^\chi(M)$ for (possibly finite) $\chi < cf\ \alpha$, and p does not fork over M, then there is some $i_0 < \alpha$ such that p does not fork over M_{i_0}*

3. *If K is categorical in some $\lambda = \lambda^{<\kappa}$, then $\downarrow$ is an independence relation with $\kappa_\alpha(\downarrow) \leq \omega + |\alpha|$.*

Proof:

1. Suppose that $A \underset{M_0}{\downarrow} N$ and let $N^+ \succ N$. In particular, this means that $A \underset{N}{\downarrow} N$ and every $< \kappa$ approximation to $tp(A/N)$ is realized in N. We can use this to construct U as in the previous chapter such that $h(tp(A/N))$ is realized in $\Pi N/U$. That means that $tp(A/N)$ is realized in N^U. Call this realization A'. By the above claim, $N^U \underset{N}{\downarrow} N^+$. By (M), this implies $A' \underset{N}{\downarrow} N^+$. Since $tp(A/N) = tp(A'/N)$, $A' \underset{M_0}{\downarrow} N$ by invariance. Thus, by (T), $A' \underset{M_0}{\downarrow} N^+$, as desired.

2. We break into cases based on the cofinality of α.

   If $\text{cf } \alpha < \kappa$, then, as before, we can use the fact that p does not fork over M to find a κ complete ultrafilter U on I such that p is realized in M^U. Since $\text{cf } \alpha < \kappa$ and U is κ complete, we have that $M^U = \cup_{i<\alpha} M_i^U$. Let $A \in M^U$ realize p. Since A if of size χ and $\chi < \text{cf } \alpha$, there is some $i_0 < \alpha$ such that $A \in M_{i_0}^U$. Thus, by the claim,

$$M_{i_0}^U \underset{M_{i_0}}{\downarrow} M$$

$$A \underset{M_{i_0}}{\downarrow} M$$

   Thus, $p = tp(A/M)$ does not fork over M_{i_0}.

   Now suppose that $\text{cf } \alpha \geq \kappa$. For contradiction, suppose that p forks over M_i for all $i < \alpha$. We now build an increasing and continuous sequence of ordinals $\langle i_j : j < \chi^+ + \aleph_0 \rangle$ by induction. Let $i_0 < \alpha$ be arbitrary. Given i_j, we know that p forks over M_{i_j}. By the definition, there is a small $M^- \prec M$ and small $I_0 \subset \chi$ such that $p^{I_0} \restriction M^-$ is not realized in M_{i_j}. Since $\text{cf } \alpha \geq \kappa$, there is some $i_{j+1} > i_j$ such that $M^- \prec M_{i_{j+1}}$. Then $p \restriction M_{i_{j+1}}$ forks over M_{i_j}. Set $M^* = \cup_{j<\chi^+} M_{i_j}$. Then, by Monotonicity, $p \restriction M^*$ forks over M_{i_j} for all $j < \chi^+$. Since $\chi^+ < \kappa$, this contradicts the first part.

3. Note that the results of the last Chapter say that this categoricity assumption also implies that $K_{\geq \kappa}$ has amalgamation, joint embedding, and no maximal models, which significantly weakens the reliance on or eliminates the need for Hypothesis 3.1.

   From inaccessibility, we know that $\sup_{\mu<\kappa} (\beth_{(2^\mu)^+}) = \kappa$, so Existence holds by Theorem 5.3.4. Then Extension holds by the first part, so (E) holds. Theorem 4.2.5 tells us that K is $< \kappa$ tame and type short. Finally, as outlined in the discussion after Theorem 5.1, the weak κ order property with κ inaccessible implies many models in all cardinals above κ, which is contradicted by categoricity in λ.   †

Additionally, with the full strength of a strongly compact cardinal, we can reprove much or all of [MaSh285].§4 in an AEC context. One complication is that Definition [MaSh285].4.23 defines weakly orthogonal types by having an element in the nonforking relation where we require a model. However, this definition has already been generalized at [Sh:h].III.6.

The last chapter also proves weaker of Theorem 4.2.5 from assumptions of measurable or weakly compact cardinals. These in turn could be used to produce weaker versions of Theorem 5.6.1. However, [MaSh285] is not the only time independence relations have been studied in infinitary contexts with large cardinals. Kolman and Shelah [KoSh362] and Shelah [Sh472] investigate the consequences of categoricity in $L_{\kappa,\omega}$ when κ is measurable. In [KoSh362], they use heavily 'suitable operations,' by which they mean taking κ complete ultralimits. The denote such an ultralimit of M by $Op(M)$ and the canonical embedding by $f_{Op} : M \to Op(M)$. In [Sh472], Shelah introduces the following independence relation.

Definition 5.6.2 ([Sh472].1.5)**.** *Let K be essentially below κ measurable. Define a 4-place relation $^S \underset{M_0}{\overset{M_3}{\downarrow}}$ by $M_1{}^S \underset{M_0}{\overset{M_3}{\downarrow}} M_2$ iff there is an ultralimit operation Op with embedding f_{Op} and $h : M_3 \to Op(M_1)$ such that the following commutes*
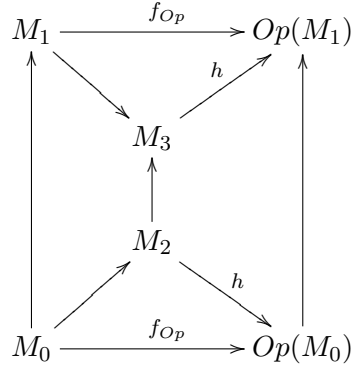
In these conditions, this notion turns out to be dual to our non-forking. Thus, by Proposition 5.4.2, it is equivalent to heir over.
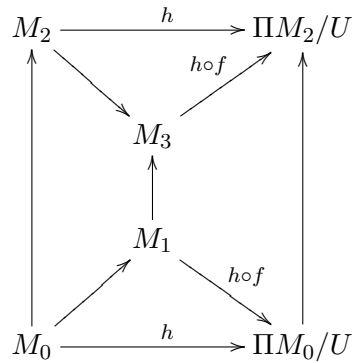
Theorem 5.6.3. *Let K be an AEC essentially below κ measurable and let $M_0 \prec M_1, M_2 \in K$. Then*

$$M_1 \underset{M_0}{\downarrow} M_2 \iff \exists M_3 \text{ so } M_2{}^S \overset{M_3}{\underset{M_0}{\downarrow}} M_1$$

Proof: First, suppose that $M_1 \underset{M_0}{\downarrow} M_2$. Then we can find a κ complete ultrafilter U such that M_0^U realizes $tp(M_1/M_2)$. Then $M_0^U \underset{M_0}{\downarrow} M_2$. Thus, there is some $f \in Aut_{M_2}\mathfrak{C}$ such that $f(M_1) \prec M_0^U$. Set $M_3 = f^{-1}[M_2^U]$. Then we have the following commuting diagram:
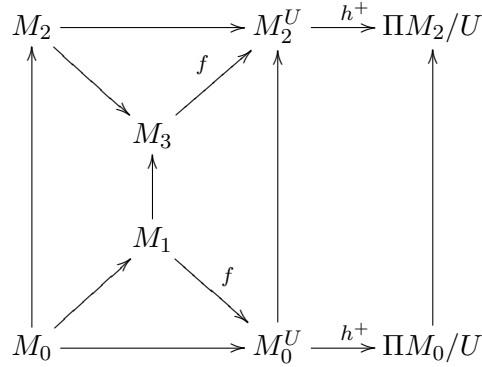
Collapsing this diagram gives

Note that an ultraproduct is a suitable ultralimit operation, and the ultrapower embedding is its corresponding embedding. Thus $M_2{}^S \overset{M_3}{\underset{M_0}{\downarrow}} M_1$.

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Second, suppose that there is an M_3 such that $M_2 {}^S \overset{M_3}{\underset{M_0}{\smile}} M_1$. The claim above generalized to ultralimits implies $f_{Op}^{-1}(Op(M_0)) \underset{M_0}{\smile} M_2$. We have that $h : M_1 \to Op(M_0)$, so by Monotonicity $f_{Op}^{-1}(h(M_1)) \underset{M_0}{\smile} M_2$. By the diagram, $f_{Op}^{-1} \circ h$ fixes M_2, we have that $tp(f_{Op}^{-1}(h(M_1))/M_2) = tp(M_1/M_2)$. By Invariance, this means that $M_1 \underset{M_0}{\smile} M_2$. $\qquad\qquad\qquad\qquad$ †

5.7 Future work

As always, new answers lead to new questions.

Based on the results that we have, a further investigation of type shortness would be useful. Because it was only defined recently, there has been no study of type shortness outside of this thesis. A starting place would be to look at known examples of AECs and determine whether or not they are type short. The relationship between tameness and type shortness explored in Chapter II suggests that the examples of [HaSh323], [BK09], and [BlSh862] would be good places to look for the failure of type shortness, while the list of tame AECs given in [GV06a] would be good candidates to prove type short.

In addition to type shortness, the above results require that the AEC be tame for long types, not just for types of length 1. Unfortunately, tameness for 1-types is the property that is typically studied. Thus, it would be interesting to see if there is some transfer theorem that shows, given $\beta < \alpha$, if tameness for β-types implies tameness for α-types or if there is some counterexample. A partial transfer theorem has recently been obtained by Boney and Vasey [BoVa] by using Shelah's good λ-frames.

A natural question to ask following the introduction of a strong independence relation in this contexts is if it is the *only* such relation, akin to first order results of Lascar for superstable theories [Las76], Harnik and Harrington for stable theories [HH84], and Kim and Pillay for simple theories [KP97]. This has been explored by Boney, Grossberg, Kolesnikov, and Vasey in [BGKV] with a positive answer:

Theorem 5.7.1 ([BGKV].7.1)**.** *Under the hypotheses of Theorem 5.3.1, $\smile$ is the only independence relation on $K_{\geq \kappa}$. In particular, if $\overset{*}{\smile}$ satisfies (I), (M), (E), and (U), then $\overset{*}{\smile} = \smile$.*

Chapter 6

Tameness and Frames

6.1 Introduction

In this chapter, we combine two recent developments in Abstract Elementary Classes (AECs): tameness and good λ-frames. Doing so allows us to extend a good λ-frame $\mathfrak{s}$, which operates only on λ-sized models, to a good frame $\geq \mathfrak{s}$ that is a forking notion for the entire class. Precisely, we prove the following.

Theorem 6.1.1. *If K is λ-tame for 1- and 2- types, $\mathfrak{s}$ is a good λ-frame, and K satisfies the amalgamation property, then $\geq \mathfrak{s}$ is a good frame. In particular, $K_{\geq\lambda}$ has no maximal models, is stable in all cardinals, and has a unique limit model in each cardinal.*

The first volume of [Sh:h], Jarden and Shelah [JrSh875], and Jarden and Sitton [JrSi13] are focused on using frames to develop classification theory for AECs. This is done by taking good λ-frames and shrinking the class as the size of the models goes up. We avoid this very complicated process by the use of tameness. Shelah defines a more general notion of an extended frame $\geq \mathfrak{s}$, but does so only as "an exercise to familiarize the reader with λ frames" [Sh:h](p. 264). He shows that some of the frame properties follow (see Theorem 6.1.3). Here we use tameness to derive the remaining properties. Note that we use the definition of frames from the more recent [JrSh875]. This definition leaves out some of the redundant clauses and, more significantly, does not require the existence of a superlimit model.

Prior to this work, there has been no work examining frames and tameness together. Hopefully, this will change. While the concepts might seem orthogonal at first glance, there is a surprising amount of interplay between them. Beyond Theorem 6.2.1, which shows Uniqueness for $\geq \mathfrak{s}$ is equivalent to λ-tameness for basic types, many aspects of frames and frame extensions rely on tameness-like locality principles and, in the other direction, many tameness results, such as categoricity transfer, rely on the concept of minimal types, which were introduced in [Sh576] and eventually turned into a frame (see [Sh:h].II.§3.7).

It should be noted that there is a loss when these two hypotheses are combined. We consider here tameness in an AEC with full amalgamation and joint embedding. These assumptions commonly appear in addition to tameness: amalgamation is used to make types well behaved and joint embedding then follows from λ-joint embedding. However, these global assumptions are in contrast to the project of frames, which aims to inductively build up a structure theory, cardinal by cardinal, and derive these properties along the way with the aid of weak diamond. On the other hand, the existence of frames in the most general setting (see [Sh:h].II.§3) uses categoricity in two successive cardinals (and more). If we add no maximal models to this hypothesis, this is already enough to apply the full categoricity transfer of [GV06a].

On the other hand, the combination of these hypotheses gives much more than just the sum of their parts. Despite the categoricity transfer results under a tameness hypothesis, there is no robust independence notion for these classes. The closest approximation is likely Chapter V of this thesis, where an independence notion of '$< \kappa$ satisfiability' is developed. Although this notion is well-behaved, additional methods beyond tameness are needed. Using these method in this paper, we have an independence notion for tame and categorical AECs under some very mild cardinal arithmetic assumptions; see Theorem 6.7.3. Looking at good λ-frames, the method for building larger frames is a complicated process that changes the Abstract Elementary Class and drops many of the models; see [Sh:h], especially II.§9.1. Although this is fine for the end goal, a process that deals with the whole class would likely have more applications. We provide such a process for tame AECs.

The sections of this chapter show that the various properties of frames extend to $\geq \mathfrak{s}$ under the assumption of tameness. They are organized so that the results only rely on the principles assumed in previous section. In particular, the stability transfer results of Section 6.3 do not rely on the tameness for 2-types assumption introduced in Section 6.5. We then discuss an application to superstability for AECs in Section 6.8 and conclude with an example in Section 6.9.

Important hypotheses are introduced at the end of Sections 8.1, 6.2, and 6.5.

Most of the citations in this chapter are from Chapter II of [Sh:h], which had previously been circulated as [Sh600]. Occasionally, we will prove a slight variation or weakening of a result from there. We denote this by adding an asterisk or minus sign, respectively, to the citation and indicate the change.

The following is Shelah's exercise in increasing the size of frames. This can be seen as a generalization of the standard technique of taking an AEC in λ and blowing it up to an AEC; recall Definition 2.1.7. We replace his notation "$\underset{<\infty}{\perp}$" with "$\underset{\geq\mathfrak{s}}{\perp}$" because it is more consistent with the notion of referring to the extended frame $\geq \mathfrak{s} = (K, \mathbf{S}_{\geq\mathfrak{s}}^{bs}, \underset{\geq\mathfrak{s}}{\perp})$.

Definition 6.1.2.

[Sh:h].II.§2.4.1) $K^{3,bs} = K_{\geq\mathfrak{s}}^{3,bs} = \{(a, M, N) \in K^{3,na}$ *:there is $M' \prec M$ from K_λ such that, for all $M'' \in K_\lambda$, $M' \prec M'' \prec M$ implies that $tp(a/M'', N) \in \mathbf{S}^{bs}(M'')$ does not fork over $M'\}$*

[Sh:h].II.§2.7/.8.1) *For $M \in K$,*

$$\mathbf{S}_{\geq s}^{bs}(M) = \{p \in \mathbf{S}(M) : \text{for some/every } tp(a/M, N) = p, (a, M, N) \in K^{3,bs} \}$$

[Sh:h].II.§2.5) *We say that $\underset{\geq\mathfrak{s}}{\perp}(M_0, M_1, a, M_3)$ holds iff $M_0 \prec M_1 \prec M_3 \in K$, $a \in M_3 - M_1$, and there is $M_0' \prec M_0$ from K_λ such that if $M_0' \prec M_1' \prec M_1$ and $M_1' \cup \{a\} \subset M_3' \prec M_3$ with $M_1', M_3' \in K_\lambda$, then $\underset{\mathfrak{s}}{\perp}(M_0', M_1', a, M_3')$.*

[Sh:h].II.§2) *If $\mathfrak{s}$ is a good λ-frame, then set $\geq \mathfrak{s} := ((K_\mathfrak{s})^{up}, \mathbf{S}_{\geq\mathfrak{s}}^{bs}, \underset{\geq\mathfrak{s}}{\perp})$.*

  1. *$\geq \mathfrak{s}$ is a* good frame *iff it satisfies the axioms for good λ-frames after removing the restriction on the size of the models and length of sequences.*

Many of the properties of good λ-frames transfer upwards immediately.

Theorem 6.1.3. *If $\mathfrak{s}$ is a good λ-frame, then $\geq \mathfrak{s}$ is a good frame, except possibly for (C), (D)(d), and (E)(e), (f), and (g).*

Proof: By the results of [Sh:h].II.§2. Specifically, Invariance and (D)(a) are 8.3, Density is 9, Monotonicity is 11.3, Transitivity is 11.4, Local Character is 11.5, and Continuity is 11.6. †

At least some additional hypothesis is necessary to transfer all properties of a good λ-frame $\mathfrak{s}$ to a good frame $\geq \mathfrak{s}$. This can be observed by observing that the Hart-Shelah examples [HaSh323] (reanalyzed more deeply by Baldwin and Kolesnikov in [BK09]) have good λ-frames at low cardinalities, but the upward extension fails Uniqueness and Basic Stability (and only those) exactly at the cardinal that tameness breaks down; see Section 6.9 for details.

In light of this, to prove that $\geq \mathfrak{s}$ is a good frame, we need to additionally show amalgamation, joint embedding, no maximal models, uniqueness, basic stability, extension existence, and symmetry. In order to avoid any mention of categoricity or non-structure arguments that require instances of the weak continuum hypothesis (as in [Sh576] or [Sh:h].I.§3), we assume amalgamation. This leads us to our first hypothesis.

Hypothesis 6.1.4. K *is an AEC with* $LS(K) \leq \lambda = \lambda_\mathfrak{s}$ *with amalgamation and* $\mathfrak{s}$ *is a good* λ*-frame.*

Although joint embedding is not included in this hypothesis, we may freely use it due to the following fact.

Fact 6.1.5. *If K is an AEC with amalgamation and K_λ has joint embedding, then $K_{\geq \lambda}$ has joint embedding.*

Additionally, Jarden and Shelah [JrSh875] introduce the notion of semi-good $\lambda_\mathfrak{s}$-frames, which replace Basic Stability with Almost Basic Stability, which requires that $|\mathbf{S}_\mathfrak{s}^{bs}(M)| \leq \lambda_\mathfrak{s}^+$ for all $M \in K_{\lambda_\mathfrak{s}}$. The following could also be done for semi-good frames, although Section 6.3 shows that, assuming tameness, $\geq \mathfrak{s}$ will be stable everywhere strictly above $\lambda_\mathfrak{s}$, even if $\mathfrak{s}$ is just a semi-good $\lambda_\mathfrak{s}$-frame.

6.2 Tameness and Uniqueness

Tameness is the key property that is necessary in extending frames, needed both for Uniqueness and Symmetry. In this section, we show that tameness for 1-types is *equivalent* to the frame having uniqueness.

We will prove the following.

Theorem 6.2.1. $K_{\geq \mathfrak{s}}$ *is $\lambda_\mathfrak{s}$-tame for basic types iff $\geq \mathfrak{s}$ satisfies Uniqueness.*

We can parametrize this result and get that $(\lambda_\mathfrak{s}, \mu)$-tameness is equivalent to Uniqueness for models of size μ. To prove this, we use and prove the following variation of a claim from Shelah's book:

Claim 6.2.2 ([Sh:h].II.§2.10*). *If $tp(a/M, N) \in \mathbf{S}_{\geq \mathfrak{s}}^{bs}(M)$, then there is some $M_0 \in K_\mathfrak{s}$ with $M_0 \prec M$ such that $tp(a/M_0, N) \in \mathbf{S}_\mathfrak{s}^{bs}(M_0)$ and*

if $M_0 \prec M' \prec M$, then $tp(a/M', N) \in \mathbf{S}_{\geq \mathfrak{s}}^{bs}(M')$ does not fork over M_0.

Our trivial variation is to permit $M' \in K_{\geq \mathfrak{s}}$, rather than restricting to the case $M' \in K_\mathfrak{s}$. We offer a proof for completeness because [Sh:h] omits one.

Proof: Since $tp(a/M, N) \in \mathbf{S}_{\geq \mathfrak{s}}^b s(M)$, we have that $(a, M, N) \in K_{\geq s}^{3,bs}$. So, by definition, there is some $M_0 \prec M$ of size $\lambda = \lambda_\mathfrak{s}$ so, for all $M'' \in K_\lambda$,

$$M_0 \prec M'' \prec M \implies \underset{\mathfrak{s}}{\downarrow}(M_0, M'', a, N)$$

Now we just need to prove (1).

Let $M' \in K$ such that $M_0 \prec M' \prec M$. First, we see that $(a, M', N) \in K_{\geq \mathfrak{s}}^{3,bs}$ as witnessed by M_0. Now we want to show that $\underset{\geq \mathfrak{s}}{\downarrow}(M_0, M', a, N)$ and, in fact, we claim that M_0 is the witness M_0' for this. If $M_1' \in K_\lambda$ such that $M_0 \prec M_1' \prec M'$, then, since $M' \prec M$, $M_1' \prec M$. Thus, by the definition of M_0 as the witness for $(a, M, N) \in K_{\geq \mathfrak{s}}^{3,bs}$, $tp(a/M_1', N) \in \mathbf{S}_\mathfrak{s}^b s(M_1')$ does not fork over M_0. So then

$\underset{\geq \mathfrak{s}}{\bigcup}(M_0, M', a, N)$ as desired. †

We define an equivalence relation $\mathcal{E}_M^{\mathfrak{s}}$, as in [Sh:h].II.§2.7.3, for $M \in K_{\geq \mathfrak{s}}$ on $\mathbf{S}^b s_{\geq \mathfrak{s}}(M)$ by $p\mathcal{E}_M^{\mathfrak{s}}q$ iff $p \restriction N = q \restriction N$ for all $N \prec M$ in $K_{\mathfrak{s}} = K_{\lambda_{\mathfrak{s}}}$.

We quote:

Fact 6.2.3 ([Sh:h].II.§2.8.5). *$\mathcal{E}_M^{\mathfrak{s}}$ is an equivalence relation on $\mathbf{S}_{\geq \mathfrak{s}}^b s(M)$ and if $p, q \in \mathbf{S}_{\geq \mathfrak{s}}^b s(M)$ do not fork over $N \in K_{\mathfrak{s}}$ such that $N \prec M$ then*

$$p\mathcal{E}_M^{\mathfrak{s}}q \iff (p \restriction N = q \restriction N)$$

Proof of Theorem 6.2.1: First, suppose that $\geq \mathfrak{s}$ satisfies Uniqueness for some $M \in K_\mu$ with $\mu \geq \lambda_{\mathfrak{s}}$. Let $p, q \in \mathbf{S}_{\geq \mathfrak{s}}^b s(M)$ such that $p \restriction N = q \restriction N$ for all $N \prec M$ of size λ. Then we can find M_p, M_q as in Claim 6.2.2 above. Let $M' \prec M$ of size λ contain both. Then by Monotonicity, we know that p and q both don't fork over M'. However, by assumption, $p \restriction M' = q \restriction M'$. Then, by Uniqueness, $p = q$.
Second, suppose that K_s is $(\lambda_{\mathfrak{s}}, \mu)$ tame for basic types. In particular, this means that $\mathcal{E}_M^{\mathfrak{s}}$ is equality for all $M \in K_\mu$. Let $M \in K_\mu$, $p, q \in \mathbf{S}_{\geq \mathfrak{s}}^b s(M)$, and $M' \prec M$ such that p and q do not fork over M' (in the sense of $\geq \mathfrak{s}$) and $q \restriction M' = q \restriction M'$. By Claim 6.2.2, there are $M_p, M_q \prec M$ of size λ such that $p \restriction M'$ does not fork over M_p and $q \restriction M'$ does not fork over M_q. As above, find $M_0 \prec M'$ of size λ to contain M_p and M_q; then by Monotonicity, $p \restriction M'$ and $q \restriction M'$ do not fork over M_0. Then by Transitivity, p and q don't fork over M_0. Also, since $M_0 \prec M'$ and $p \restriction M' = q \restriction M'$, we have $p \restriction M_0 = q \restriction M_0$. Then, by [Sh:h].II.§2.8.5, $p\mathcal{E}_M^{\mathfrak{s}}q$. But, by tameness, this is equality, so $p = q$. †

Additionally, if $\mathfrak{s}$ is type full ($S_{\mathfrak{s}}^{bs} = S^{na}$), then we can extend our result on tameness to not mentioning basic types at all.

Corollary 6.2.4. *If $\geq \mathfrak{s}$ is a type full good frame, then $K_{\geq \mathfrak{s}}$ is $\lambda_{\mathfrak{s}}$ tame.*

Note that [Sh:h].II.§6.36 says that we can assume $\mathfrak{s}$ is full if it has existence for $K_{\lambda_{\mathfrak{s}}}^{3,uq}$ (see [Sh:h].II.§5.3). In light of these results, we add the following hypothesis.

Hypothesis 6.2.5. *K is $\lambda_{\mathfrak{s}}$-tame for basic 1-types.*

6.3 Basic Stability

In this section, we use only tameness for 1-types (and therefore no symmetry) to prove that an extended frame leads to basic stability in all larger cardinals. This is similar to the first order argument that stability and $\kappa(T) = \omega$ together imply superstability. This has been done in non-elementary contexts by Makkai and Shelah [MaSh285].4.14.

Theorem 6.3.1. *For all $\kappa \geq \lambda$, K is κ-stable for $\geq \mathfrak{s}$ basic types; that is, for all $M \in K_\kappa$, $|\mathbf{S}_{\geq \mathfrak{s}}^{bs}(M)| \leq \kappa$. In particular, $(\lambda, \leq \kappa)$-tameness for basic 1-types implies κ-stability for basic types.*

Proof: We proceed by induction on $\lambda \leq \mu \leq \kappa$. If $\mu = \lambda$, then this is the hypothesis. For $\mu > \lambda$, let $M \in K_\mu$ and find a resolution $\langle M_i \in K_{[\lambda,\mu)} : i < \mathrm{cf}\,\mu \rangle$ of M. By Local Character for $\geq \mathfrak{s}$, for each $p \in S_{\geq \mathfrak{s}}^{bs}(M)$, there is some $i_p < \mathrm{cf}\,\mu$ such that p does not fork over M_{i_p} in the sense of $\geq \mathfrak{s}$. By Theorem

6.2.1, $\geq_\mathfrak{s}$ satisfies Uniqueness for domains of size μ, so the map $p \mapsto M_{i_p}$ is injective from $\mathbf{S}^{bs}_{\geq_\mathfrak{s}}(M)$ to $\bigcup_{i < \mathrm{cf}\, \mu} \mathbf{S}^{bs}_{\geq_\mathfrak{s}}(M)$. So

$$|\mathbf{S}^{bs}_{\geq_\mathfrak{s}}(M)| \leq \sum_{i < \mathrm{cf}\, \mu} |\mathbf{S}^{bs}_{\geq_\mathfrak{s}}(M_i)| = \mu$$

$$\dagger$$

In particular, this uses only Local Character and Uniqueness. We can extend this to full stability using [Sh:h].II.§.4.2.1, which shows that stability for basic types implies stability for all types using amalgamation, Density, Monotonicity, and Local Character. Thus, we get the following stability transfer which improves on results of Grossberg and VanDieren [GV06b]; Baldwin, Kueker, and VanDieren [BKV06]; and Lieberman [Lie13], but adds the assumption of a good λ-frame.

Corollary 6.3.2. *Suppose K is χ-tame for 1-types and has a good χ-frame except possibly for the assumption of basic stability. If K is stable (or just stable for basic types) in some $\kappa \geq \chi$, then it is stable in all $\mu \geq \kappa$.*

6.4   Extension Existence

We now turn to the existence of nonforking extensions of basic types. One of the difficulties of using Galois types (compared to syntactic types) is that an increasing sequence of types need not have an upper bound. Shelah and Baldwin [BlSh862].3.3 construct an example of an AEC that has an increasing sequence of types with no upper bound from $2^{\aleph_0} = \aleph_1$, $\diamondsuit_{\aleph_1}$, $\diamondsuit_{S^{\aleph_2}_{\mathrm{cf}\, \aleph_1}}$, and $\square_{\aleph_2}$. However, if we require that the sequence is *coherent* (see below), then there is an upper bound. Equivalently, Shelah [Sh576] and others work with increasing sequences from $K^{3,na}_\lambda$.

In essence, we will show that a good λ-frame and λ-tameness imply that types are local and apply an argument similar to [Sh394] (proved as [Bal09].11.5) to show that compactness follows; see [BlSh862] for the relevant definitions, although we will not use them here. This is essentially the same argument used in the proof of [GV06a].2.22, where they work with quasiminimal types. In all cases, there is some property–locality, quasiminimality, or Uniqueness–that ensures that there is only one possible extension at limit steps. We reprove this here because previous contexts have worked inside of a monster model, which we do not have. However, the ideas in Proposition 6.4.2 are not new.

Definition 6.4.1. *Given increasing sequences $\langle M_i : i < \delta \rangle$ and $\langle p_i \in \mathbf{S}(M_i) : i < \delta \rangle$, the sequence of types is called* coherent *iff there are, for $j < i < \delta$, models N_i, elements a_i, and maps $f_{j,i} : N_j \to N_i$ so*

1. *for all $k < j < i < \delta$, we have $f_{k,i} = f_{j,i} \circ f_{k,j}$;*

2. *$tp(a_i/M_i, N_i) = p_i$;*

3. *$f_{j,i}$ fixes M_j for all $i > j$; and*

4. *$f_{j,i}(a_j) = a_i$.*

If we have a coherent sequence of types, it must have an upper bound. Namely, taking $M = \bigcup_{i < \delta} M_i$, $(N, f_i^*)_{i < \delta} = \varinjlim_{j < k < \delta}(N_k, f_{j,k})$, and $a^* = f_0^*(a_0)$, the upper bound is $tp(a^*/M, N)$.

The above does not require frames. However, if we have a frame, then all nonforking sequences of types are coherent.

Proposition 6.4.2. *Let $\langle M_i \in K_{\geq \lambda_{\mathfrak{s}}} : i < \delta \rangle$ be an increasing, continuous sequence. If $\langle p_i \in \mathbf{S}^{bs}_{\geq \mathfrak{s}}(M_i) : i < \delta \rangle$ is an increasing sequence of basic 1-types such that each p_i does not fork over M_0, then p_i is coherent. Thus, there is $p_\delta \in S^{bs}_{\geq \delta}(M_\delta)$ extending each p_i.*

Note that Uniqueness (which follows from Theorem 6.2.1 since we assumed tameness for basic types in Hypothesis 6.2.5) is the key property used in this proof.

Proof: For $i = 0$, set $(a_0, M_0, N_0) \in K^{3,bs}$ to be some triple realizing p_0.

For i limit, set $(N_i, f_{j,i})_{j < i} = \varinjlim_{l < k < i}(N_k, f_{l,k})$. Then $M_i \prec N_i$. Set $a_i = f_{0,i}(a_i)$, which is equal to $f_{j,i}(a_j)$ for any $j < i$. For each $j < i$, $f_{j,i}$ fixes M_j, so $a_i \vDash p_j$. Thus, $tp(a_i/M_j, N_i)$ doesn't fork over M_0. Since this is true for all $j < i$, Continuity says that $tp(a_i/M_i, N_i)$ does not fork over M_0. Since p_i also does not fork over M_0, Uniqueness implies that $tp(a_i/M_i, N_i) = p_i$, as desired.

For $i = j + 1$, find (a_i', M_i, N_i') such that $tp(a_i'/M_i, N_i') = p_i$. Since $p_i \restriction M_j = p_j$, a_i and a_j realize the same type over M_i. Thus we can construct the following commutative diagram

so $f_{j,i}(a_j) = a_i$. Then set $f_{k,i} = f_{j,i} \circ f_{k,j}$ for any $k < j$.

Once we have constructed the coherent sequence, there is some $p \in \mathbf{S}(M)$ for $M = \cup_{i < \delta} M_i$ that extends each p_i. By Continuity, $p \in \mathbf{S}^{bs}_{\geq \mathfrak{s}}(M)$ and p does not fork over M_0. †

Now we prove that Extension Existence holds in $\geq \mathfrak{s}$. We proceed by induction.

Theorem 6.4.3. $\geq \mathfrak{s}$ *satisfies Axiom (E)(g).*

Proof: We want to show:

If $M \prec N$ from $K_{\geq \lambda_{\mathfrak{s}}}$ and $p \in \mathbf{S}^b_{\geq \mathfrak{s}}s(M)$, then there is some $q \in \mathbf{S}^b_{\geq \mathfrak{s}}s(N)$ such that $p \leq q$ and q does not fork over M (in the $\underset{\geq \mathfrak{s}}{\downarrow}$ sense).

We will prove this by induction on $\|N\|$.

Base Case: $\|N\| = \lambda_{\mathfrak{s}}$

Then $\|M\| = \lambda_{\mathfrak{s}}$ as well, and this follows from $\mathfrak{s} = (\geq \mathfrak{s}) \restriction \lambda_{\mathfrak{s}}$ being a good $\lambda_{\mathfrak{s}}$-frame.

Inductive Step: $\|N\| = \mu > \lambda_{\mathfrak{s}}$

We break into two cases based on the size of M.

If $\|M\| < \|N\|$, then we find a resolution $\langle N_i \in K_{<\mu} \mid i < \mu \rangle$ such that $N_0 = M$. By induction, we will construct increasing $p_i \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N_i)$ such that p_i does not fork over N_0 and extends p. Clearly, $p_0 = p$.

For i limit, by Proposition 6.4.2, we can find some p_i such that $p_i \restriction N_j = p_j$ for all $j < i$. Then $p_i \restriction N_j$ does not fork over M for all $j < i$, so, by Continuity, p_i does not fork over M.

For $i = j + 1$, we use our induction to extend p_j to some $p_i \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N_i)$ that doesn't fork over N_j; this is valid since $\|N_i\| < \|N\|$.

Then, we use Proposition 6.4.2 a final time to find $q \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N)$ such that $q \restriction N_i = p_i$. By Continuity, this means q does not fork over M as desired.

If $\|M\| = \|N\|$, we find $M_0 \prec M$ in $K_{\mathfrak{s}}$ as in Claim 6.2.2 such that if $M_0 \prec M' \prec M$, $p \restriction M'$ does not fork over M_0. Then we use this as the start for a resolution $\langle M_i \in K_{<\mu} \mid i < \operatorname{cf} \mu \rangle$ of M. Set $p_i = p \restriction M_i$; note that p_i does not fork over M_0. Now we find a resolution $\langle N_i \in K_{<\mu} \mid i < \operatorname{cf} \mu \rangle$ of N such that $M_i \prec N_i$. We are going to find increasing $q_i \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N_i)$ by induction such that q_i does not fork over M_0 and $p_i \leq q_i$.

We use the induction hypothesis to find $q_0 \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N)$ that extends p_0 and does not fork over M_0.

For i limit, use the induction hypothesis to find $q_i \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N_i)$ that extends all q_i. By continuity, q_i does not fork over M or over N_j for all $j < i$.

For $i = j + 1$, use induction to find $q_i \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N_i)$ such that $q_i \geq q_j$ and q_i does not fork over N_j. Then, by Transitivity, q_i does not fork over M_0. Also note that p_i does not fork over M_0 and $q_i \restriction M_0 = p_0 = p_i \restriction M_0$, so Uniqueness tells us that $q_i \restriction M_i = p_i \restriction M_i = p_i$.

Now we use Proposition 6.4.2 to set $q \in \mathbf{S}^b_{\geq \mathfrak{s}} s(N)$ to extend all q_i and p_0. Again by Continuity, q does not fork over M_0. Also, $q \restriction M_0 = p_0 = p \restriction M_0$ so, since p also does not fork over M_0, we can use Uniqueness to get that $q \restriction M = p$. Finally, by Monotonicity, we have that q does not fork over M. †

6.5 Tameness and Symmetry

In this section, we show that tameness for 2-types implies Symmetry in $\geq \mathfrak{s}$. Unfortunately, unlike Section 6.2, this is not shown to be an equivalence. This is enough for our goal of extending a frame, but a characterization of exactly when Symmetry holds in $\geq \mathfrak{s}$ would be better. We know that tameness for 2-types (or even tameness for basic 2-types in the sense of [Sh:h].III.§5.2) is not this characterization because the Hart-Shelah examples have frames with Symmetry at all cardinals, including after the tameness fails; see Section 6.9. Additionally, the precise relationship between tameness for 1-types and tameness for 2-types is not currently known, although tameness for 2-types clearly implies tameness for 1-types.

Theorem 6.5.1. *If K satisfies $\lambda_{\mathfrak{s}}$ tameness for 2-types, then $\geq \mathfrak{s}$ satisfies Axiom (E)(f).*

For reference, a diagram of the models involved in the proof is included in the proof. This diagram and the naming convention for models deserves some explanation and we are indebted to the referee for pushing us to a better presentation. Functions like f and g *above* arrows have their usual meanings (that f

is a K embedding between models), but we write elements *under* arrows to indicate that the element is in the end model but not the starting model.

The majority of models are names $M[i, j, \chi]$ for i and j natural numbers and χ a cardinal, either μ or $\lambda_\mathfrak{s}$. The cardinal χ denotes the size of the model and the sizes separate the models into two levels. If we have $M[i, j, \chi]$ and $M[i', j', \chi']$ with $i \leq i'$, $j \leq j'$, and $\chi \leq \chi'$, then this will mean that $M[i, j, \chi]$ is embedded into $M[i', j', \chi']$ by the map indicated by the diagram. In particular, we *do not* have an embedding of $M[4, 2, \lambda_\mathfrak{s}]$ into $M[3, 1, \mu]$, even though the first model is below the second in the diagram (and the nonstandard indices for the first model are picked to indicate this). The exception to this convention are the models M^-, M_I, and M_{II}. These models are all contained in $M[0, 0, \lambda_\mathfrak{s}]$ and are used as "helper models;" that is, they lend properties to $M[0, 0, \lambda_\mathfrak{s}]$, but are not directly used in the proof.

Proof: Suppose we have $M[0, 0, \mu], M[0, 1, \mu], M[1, 1, \mu] \in K_\mu$ such that $\underset{\geq \mathfrak{s}}{\perp}(M[0, 0, \mu], M[0, 1, \mu], a_1, M[1, 1, \mu]$ and $a_2 \in M[0, 1, \mu]$ such that $tp(a_2/M[0, 0, \mu], M[1, 1, \mu]) \in \mathbf{S}^{bs}_{\geq \mathfrak{s}}(M[0, 0, \mu])$. Let $M[1, 0, \mu] \in K_\mu$ such that $M[0, 0, \mu] \prec M[1, 0, \mu] \prec M[1, 1, \mu]$ and $a_1 \in M[1, 0, \mu]$. By Extension Existence, there is $M[2, 1, \mu] \in K_\mu$ such that $M[1, 1, \mu] \prec M[2, 1, \mu]$ and $a' \in M[2, 1, \mu]$ such that $\underset{\geq \mathfrak{s}}{\perp}(M[0, 0, \mu], M[1, 0, \mu], a', M[2, 1,$ and $tp(a'/M[0, 0, \mu], M[2, 1, \mu]) = tp(a_2/M[0, 0, \mu], M[1, 1, \mu])$. We want to show that this type equality still holds if we add a_1.

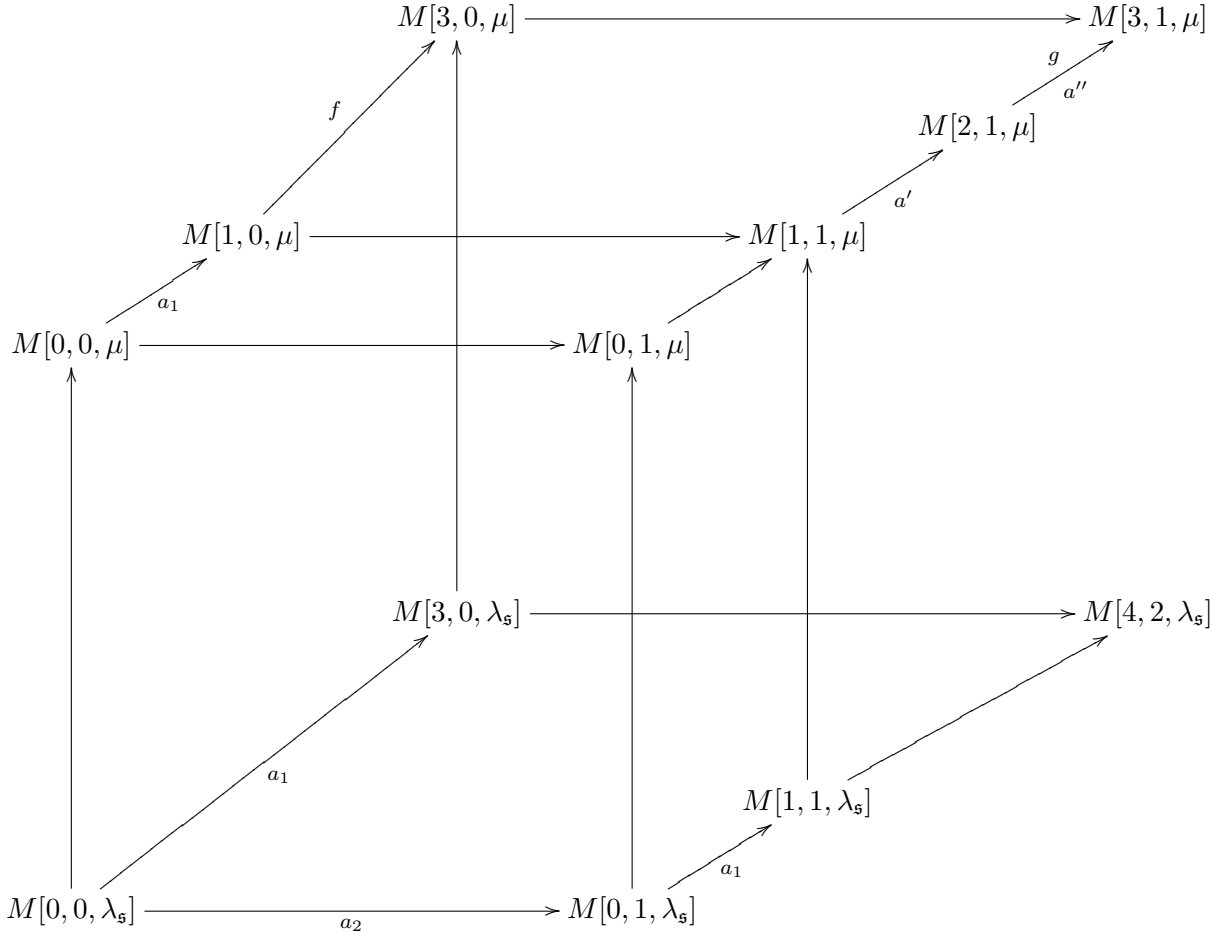**Main Claim:** $tp(a_1 a_2/M[0, 0, \mu], M[1, 1, \mu]) = tp(a_1 a'/M[0, 0, \mu], M[2, 1, \mu])$

This is Enough: Let $N \in K_\mu$ witness the above type equality; that is, $M[1, 1, \mu] \prec N$ and there is $f : M[2, 1, \mu] \rightarrow_{M[0, 0, \mu]} N$ such that $f(a_1 a') = a_1 a_2$. Then apply f to $\underset{\geq \mathfrak{s}}{\perp}(M[0, 0, \mu], M[1, 0, \mu], a', M[2, 1, \mu])$; this shows that $\underset{\geq \mathfrak{s}}{\perp}(M[0, 0, \mu], f(M[1, 0, \mu]), a_2, N)$. This proves Symmetry since $a_1 \in f(M[1, 0, \mu])$.

Proof of Main Claim: Fix $M^- \prec M[0, 0, \mu]$ of size $\lambda_\mathfrak{s}$. From the assumption of tameness for 2-types, it suffices to show

$$tp(a_1 a_2/M^-/M[1, 1, \mu]) = tp(a_1 a'/M^-, M[2, 1, \mu])$$

By the definition of $\geq \mathfrak{s}$, there are $M_I, M_{II} \in K_{\lambda_\mathfrak{s}}$ such that $M_I, M_{II} \prec M[0, 0, \mu]$ and that witness (in the sense of the definition of $\perp$, see Definition 6.1.2) $\underset{\geq \mathfrak{s}}{\perp}(M[0, 0, \mu], M[0, 1, \mu], a_1, M[1, 1, \mu])$ and $\underset{\geq \mathfrak{s}}{\perp}(M[0, 0, \mu], M[1, 0, \mu], a', M[2, 1, \mu])$, respectively. Let $M[0, 0, \lambda_\mathfrak{s}] \in K_{\lambda_\mathfrak{s}}$ such that $M[0, 0, 0]_{\lambda_\mathfrak{s}} \prec M[0, 0, \mu]$ and it contains M^-, M_I, and M_{II}. Then, since $M[0, 0, \lambda_\mathfrak{s}]$ contains witnesses to the nonforking, we have that

1. if there are $M, M' \in K_{\lambda_\mathfrak{s}}$ with $a_1 \in M'$ such that $M[0, 0, \lambda_\mathfrak{s}] \prec M \prec M[0, 1, \mu]$ and $M \prec M' \prec M[1, 1, \mu]$, then $\underset{\mathfrak{s}}{\perp}(M[0, 0, \lambda_\mathfrak{s}], M, a_1, M')$; and

2. if there are $M, M' \in K_{\lambda_\mathfrak{s}}$ with $a' \in M'$ such that $M[0, 0, \lambda_\mathfrak{s}] \prec M \prec M[1, 0, \mu]$ and $M \prec M' \prec M[2, 1, \mu]$, then $\underset{\mathfrak{s}}{\perp}(M[0, 0, \lambda_\mathfrak{s}], M, a', M')$.

$M[3,0,\mu] \longrightarrow M[3,1,\mu]$

g

a''

$M[2,1,\mu]$

f

$M[1,0,\mu] \longrightarrow M[1,1,\mu]$

a'

a_1

$M[0,0,\mu] \longrightarrow M[0,1,\mu]$

$M[3,0,\lambda_\mathfrak{s}] \longrightarrow M[4,2,\lambda_\mathfrak{s}]$

a_1

$M[1,1,\lambda_\mathfrak{s}]$

a_1

$M[0,0,\lambda_\mathfrak{s}] \xrightarrow{\quad a_2 \quad} M[0,1,\lambda_\mathfrak{s}]$

Since $\lambda_\mathfrak{s} \geq LS(K)$, there are $M[0,1,\lambda_\mathfrak{s}], M[1,1,\lambda_\mathfrak{s}] \in K_{\lambda_\mathfrak{s}}$ such that $M[0,0,\lambda_\mathfrak{s}] \prec M[0,1,\lambda_\mathfrak{s}] \prec M[0,1,\mu]$ and $a_2 \in M[0,1,\lambda_\mathfrak{s}]$; and $M[0,1,\lambda_\mathfrak{s}] \prec M[1,1,\lambda_\mathfrak{s}] \prec M[1,1,\mu]$ and $a_1 \in M[1,1,\lambda_\mathfrak{s}]$. From the definition of $M[0,0,\lambda_\mathfrak{s}]$, this implies $\underset{\geq \mathfrak{s}}{\bigsmile}(M[0,0,\lambda_\mathfrak{s}], M[0,1,\lambda_\mathfrak{s}], a_1, M[1,1,\lambda_\mathfrak{s}])$. Since Symmetry for $\mathfrak{s}$ holds, there are $M[3,0,\lambda_\mathfrak{s}], M[4,2,\lambda_\mathfrak{s}] \in K_{\lambda_\mathfrak{s}}$ such that $M[0,0,\lambda_\mathfrak{s}] \prec M[3,0,\lambda_\mathfrak{s}] \prec M[4,2,\lambda_\mathfrak{s}]$ and $M[1,1,\lambda_\mathfrak{s}] \prec M[4,2,\lambda_\mathfrak{s}]$ with $a_1 \in M[3,0,\lambda_\mathfrak{s}]$ and $\underset{\geq \mathfrak{s}}{\bigsmile}(M[0,0,\lambda_\mathfrak{s}], M[3,0,\lambda_\mathfrak{s}], a_2, M[4,2,\lambda_\mathfrak{s}])$. By chasing diagrams, $tp(a_1/M[0,0,\lambda_\mathfrak{s}], M[1,0,\mu]) = tp(a_1/M[0,0,\lambda_\mathfrak{s}], M[3,0,\lambda_\mathfrak{s}])$, so there are $M[3,0,\mu] \in K_\mu$ and $f : M[1,0,\mu] \to_{M[0,0,\lambda_\mathfrak{s}]} M[3,0,\mu]$ such that $M[3,0,\lambda_\mathfrak{s}] \prec M[3,0,\mu]$ and $f(a_1) = a_1$. Since $\geq \mathfrak{s}$ satisfies Extension Existence and K has the amalgamation property, there is a nonforking extension of $f(tp(a'/M[1,0,\mu], M[2,1,\mu]))$ to $M[3,0,\mu]$. This means that there are $M[3,1,\mu] \in K_\mu$, $a'' \in M[3,1,\mu]$, and $g : M[2,1,\mu] \to M[3,1,\mu]$ such that

- $M[3,0,\mu] \prec M[3,1,\mu]$;

- $f \subset g$;

- $tp(a''/f(M[1,0,\mu]), M[3,1,\mu]) = tp(g(a')/f(M[1,0,\mu]), M[3,1,\mu])$; and

- $\underset{\geq \mathfrak{s}}{\bigsmile}(f(M[1,0,\mu]), M[3,0,\mu], a'', M[3,1,\mu])$

Extend g to an $L(K)$-isomorphism G with range including $M[3,1,\mu]$.

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Then $\underset{\geq \mathfrak{s}}{\perp}(M[1,0,\mu], G^{-1}(M[3,0,\mu]), G^{-1}(a''), G^{-1}(M[3,1,\mu]))$ and $tp(G^{-1}(a'')/M[1,0,\mu], G^{-1}(M[3,1,\mu]))$ $tp(a'/M[1,0,\mu], M[2,1,\mu])$. Since $\underset{\geq \mathfrak{s}}{\perp}(M[0,0,\lambda_\mathfrak{s}], M[1,0,\mu], a', M[2,1,\mu])$, this type equality means that $\underset{\geq \mathfrak{s}}{\perp}(M[0,0,\lambda_\mathfrak{s}], M[1,0,\mu], G^{-1}(a''), G^{-1}(M[3,1,\mu]))$. Since $\geq \mathfrak{s}$ satisfies Transitivity, we have $\underset{\geq \mathfrak{s}}{\perp}(M[0,0,\lambda_\mathfrak{s}], G^{-1}(M[3,0,\mu]), G^{-1}(a''), G^{-1}(M[3,1,\mu]))$. Since $G \supset g \supset f$ fixes $M[0,0,\lambda_\mathfrak{s}]$ and $\geq \mathfrak{s}$ satisfies Invariance, we have $\underset{\geq \mathfrak{s}}{\perp}(M[0,0,\lambda_\mathfrak{s}], M[3,0,\mu], a'', M[3,1,\mu])$. By Monotonicity, we have $\underset{\geq \mathfrak{s}}{\perp}(M[0,0,\lambda_\mathfrak{s}], M[3,0,\lambda_\mathfrak{s}], a'', M[3,1,\mu])$. Recall that we picked $M[3,0,\lambda_\mathfrak{s}]$ such that $\underset{\geq \mathfrak{s}}{\perp}(M[0,0,\lambda_\mathfrak{s}], M[3,0,\lambda_\mathfrak{s}], a_2, M[4,2,\lambda_\mathfrak{s}])$ and that

$$
\begin{aligned}
tp(a_2/M[0,0,\lambda_\mathfrak{s}], M[4,2,\lambda_\mathfrak{s}]) &= tp(a'/M[0,0,\lambda_\mathfrak{s}], M[2,1,\mu]) \\
&= tp(g(a')/M[0,0,\lambda_\mathfrak{s}], M[3,1,\mu]) \\
&= tp(a''/M[0,0,\lambda_\mathfrak{s}], M[3,1,\mu])
\end{aligned}
$$

since g fixes $M[0,0,\lambda_\mathfrak{s}]$. By Uniqueness, $tp(a_2/M[3,0,\lambda_\mathfrak{s}], M[4,2,\lambda_\mathfrak{s}]) = tp(a''/M[3,0,\lambda_\mathfrak{s}], M[3,1,\mu])$. Since $a_1 \in M[3,0,\lambda_\mathfrak{s}]$ and $M^- \prec M[0,0,\lambda_\mathfrak{s}] \prec M[3,0,\lambda_\mathfrak{s}]$, this implies $tp(a_1 a_2/M^-, M[4,2,\lambda_\mathfrak{s}]) = tp(a_1 a''/M^-, M[3,1,\mu])$.
On the other hand, since $f(a_1) = a_1$ and f fixes $M[0,0,\lambda_\mathfrak{s}]$, we have that

$$
\begin{aligned}
tp(a''/f(M[1,0,\mu]), M[3,1,\mu]) &= tp(g(a')/f(M[1,0,\mu]), M[3,1,\mu]) \\
tp(a_1 a''/f(M[1,0,\mu]), M[3,1,\mu]) &= tp(a_1 g(a')/f(M[1,0,\mu]), M[3,1,\mu]) \\
tp(a_1 a''/M^-, M[3,1,\mu]) &= tp(a_1 g(a')/M^-, M[3,1,\mu]) = tp(a_1 a'/M^-, M[2,1,\mu])
\end{aligned}
$$

So $tp(a_1 a_2/M^-, M[1,1,\mu]) = tp(a_1 a'/M^-, M[2,1,\mu])$, as desired.
Since $M^- \prec M[0,0,\mu]$ of size $\lambda_\mathfrak{s}$ was arbitrary and K is $\lambda_\mathfrak{s}$-tame for 2-types, we have $tp(a_1 a_2/M[0,0,\mu], M[1,1,\mu])$ $tp(a_1 a'/M[0,0,\mu], M[2,1,\mu])$. This proves the claim and the theorem. $\qquad\dagger$

Thus, we add the following hypothesis. Note that basic types are only defined for types of length one, so a hypothesis of "tameness for basic 2-types" would not make sense.

Hypothesis 6.5.2. *K is $\lambda_\mathfrak{s}$-tame for 2-types*

We focus on this method for obtaining Symmetry due to its similarity to Hypothesis 6.2.5. However, there is another way to derive Symmetry that does not rely on the structure of extending the frame $\mathfrak{s}$. Recall from Shelah [Sh576] that a type $p \in S(M)$ is *minimal* iff it has at most one non-algebraic extension to any $N \succ M$ with $\|N\| = \|M\|$ and that basic types in the frame from Theorem 2.3.7 are exactly the rooted minimal types. Then [Sh:h].II.§.3.7 combines the minimality of basic types with disjoint amalgamation in $\lambda_\mathfrak{s}$ to derive Symmetry for $\mathfrak{s}$. This proof can be adapted to get the following.

Theorem 6.5.3 (Without Hypothesis 6.5.2)**.** *If basic types for $\mathfrak{s}$ are minimal and $K_{\geq \lambda_\mathfrak{s}}$ satisfies disjoint amalgamation, then $\geq \mathfrak{s}$ satisfies Axiom (E)(f).*

6.6 No Maximal Models

Recall that we are working under Hypotheses 6.1.4, 6.2.5, and 6.5.2; these are that $\mathfrak{s}$ is a good λ-frame and $K_{\geq \lambda}$ has amalgamation; that K is λ-tame for basic 1-types (in the sense of $\geq \mathfrak{s}$); and that K is λ-tame for 2-types. The results so far have shown that $\geq \mathfrak{s}$ is a good frame except possibly for the "no maximal models" clause.

In this section, we adapt the proof of [Sh:h].II.§4.13.3 to show that if $K_{\geq \kappa}$ has a good frame $\geq \mathfrak{s}$, then $K_{\geq \kappa}$ has no maximal model. This is no real change in the proof, except to include the case of where the size of the model is a limit cardinal. This proof makes use of a strengthening of Non-Forking Amalgamation that Shelah calls Long Non-Forking Amalgamation. We include a proof of the final result, which combines the work of [Sh:h].II.§4.9.1, .12.1, and .13.3, to show all of the details.

Theorem 6.6.1 ([Sh:h].II.§4.13.3*)**.** *Assume $\lambda < \kappa$ and K_κ is non-empty. Then K_κ has no maximal models.*

Proof: Let $N^0 \in K_\kappa$ and let $\langle N_i^0 \in K_{[\lambda,\kappa)} : i < \kappa \rangle$ be a resolution. From Density, we know that, for each $i < \kappa$, there is some $a_i \in N_{i+1}^0 - N_i^0$ such that $tp(a_i/N_i^0, N_{i+1}^0) \in \mathbf{S}_{\geq \mathfrak{s}}^{bs}(N_i^0)$ and some $p \in tp(b/N_0^0, N_0^1) \in \mathbf{S}_{\geq \mathfrak{s}}^{bs}(N_0^0)$; we might have $a_0 = b$ and $N_1^0 = N_0^0$, but this is okay. We will construct, by induction on $\alpha \leq \lambda$, a coherent sequence $\langle N_\alpha^1, f_{\beta,\alpha} : N_\beta \to N_\alpha \mid \beta < \alpha \leq \lambda \rangle$ such that

1. $N_\alpha^0 \prec N_\alpha^1$;

2. $\underset{\geq \mathfrak{s}}{\downarrow}(N_\alpha^0, N_{\alpha+1}^0, f_{0,\alpha+1}(b), N_{\alpha+1}^1)$; and

3. $f_0 = \mathrm{id}_{N_0^1}$.

$\alpha = 0$ is already defined. For α limit, we take a direct limit. For $\alpha = \beta + 1$, we have that $N_\alpha^0 \prec N_\alpha^1, N_{\alpha+1}^0$ with $tp(a_\alpha/N_\alpha^0, N_{\alpha+1}^0), tp(f_{0,\beta}(b)/N_\alpha^0, N_\alpha^1) \in \mathbf{S}_{\geq \mathfrak{s}^-}^{bs}(N_\alpha^0)$. Then we use Non-Forking Amalgamation to find $f_\beta : N_\beta^1 \to N_\alpha^1$ with $N_\alpha^0 \prec N_\alpha^1$ so $\underset{\geq \mathfrak{s}}{\downarrow}(N_\beta^0, N_\alpha^0, f_\beta(f_{0,\beta}(b)), N_\alpha^1)$ and $\underset{\geq \mathfrak{s}}{\downarrow}(N_\alpha^0, f_\beta(N_\beta^1), a_\alpha, N_\alpha^1)$. For $\gamma \leq \beta$, set $f_{\gamma,\alpha} = f_\beta \circ f_{\gamma,\beta}$.

This completes our construction. Now we have that $N^0 = \bigcup_{\alpha < \lambda} N_\alpha^0 \not\succeq \bigcup_{\alpha < \lambda} N_\alpha^1 = N^1 \in K_\lambda$, since $f_{0,\lambda}(b) \notin N^0$. Since $N^0 \in K_\lambda$ was arbitrary, we are done. †

This allows us to prove the existence of arbitrarily large models.

Corollary 6.6.2. *K has no maximal models. In particular, it has models of all cardinalities.*

6.7 Good Frames

We drop the previous hypotheses for this section, although K will always be an AEC.

We combine our previous results into the following theorem.

Theorem 6.7.1. *Suppose K is an AEC with amalgamation. If K has a good λ-frame $\mathfrak{s}$ and is $\lambda_\mathfrak{s}$-tame for 1- and 2- types, then $\geq \mathfrak{s}$ is a good frame.*

Proof: From Theorem 6.1.3, we know that $\geq \mathfrak{s}$ satisfies all of the axioms of a good frame except for amalgamation, joint embedding, no maximal models, uniqueness, basic stability, extension existence, and symmetry. Amalgamation and joint embedding follow from the assumption of this theorem. Uniqueness, basic stability, and extension existence follow from tameness for 1-types by Theorem 6.2.1, Corollary 6.2.1, and Theorem 6.4.3. Symmetry follows from tameness for 2-types by Theorem 6.5.1. Finally, no maximal models follows from tameness for 1- and 2-types by Corollary 6.6.2. †

This is the main theorem promised in the introduction. We provide proofs of some of the other claims as well. First, we can trade the assumption of no maximal models in the categoricity transfer of [GV06a] for a set-theoretic assumption, a slight increase in tameness, and an extra categoricity cardinal.

Theorem 6.7.2. *Let K be an AEC with amalgamation and $LS(K) < \kappa \leq \lambda$ such that*

1. *K is κ tame for 1- and 2-types; and*

2. *K is categorical in λ and λ^+ with*

    (∗) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $WDmId(\lambda^+)$ is not λ^{++}-saturated.

Then K is categorical in all $\mu \geq \lambda$.

Proof: By 2. of the hypothesis and Theorem 2.3.7, K has a good λ^+-frame $\mathfrak{s}$. By Theorem 6.7.1 and tameness, $\geq \mathfrak{s}$ is a good frame. In particular, K has no maximal models. Then, we can apply the categoricity transfer of [GV06a] to show that K is categorical in all $\mu \geq \lambda^+$ and we have $\mu = \lambda$ as part of the hypothesis. †

All in all, this is not a very good trade. On the other hand, during this proof we constructed our promised independence relation in a tame and categorical AEC. There are two related sets of assumptions that allow us to do so, both of which utilize the work of Shelah, Grossberg and VanDieren, and Theorem 6.7.1.

Proposition 6.7.3. *Let K be an AEC with amalgamation that is κ-tame for 1- and 2- types and is categorical in λ^+ with $\lambda > \kappa > LS(K)$. If either of the two following hold*

1. *K has no maximal models and joint embedding and there is some $\mu \geq \min\{\lambda^+, \beth_\chi\}$ for $\chi = (2^{\beth_{(2^{LS(K)})^+}})^+$ such that $2^\mu < 2^{\mu^+} < 2^{\mu^{++}}$ and $WDmId(\mu^+)$ is not μ^{++}-saturated; or*

2. *K is categorical in λ and $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $WDmId(\lambda^+)$ is not λ^{++}-saturated;*

then there is a good frame $\geq \mathfrak{s}$ with $\lambda_{\mathfrak{s}} = \mu^+$ in case (a) and $\lambda_{\mathfrak{s}} = \lambda^+$ in case (b).

Proof: Case (b) was handled in Theorem 6.7.2 above. In case (a), the assumption of joint embedding and no maximal models means that we can use the results of [GV06a] and [Sh394] to conclude that K is categorical in every cardinal above $\min\{\lambda^+, \beth_{(2^{\beth_{(2^{LS(K)})^+}})^+}\}$; in particular, μ and μ^+. Then we can use Theorem 2.3.7 to derive a good μ^+ frame $\mathfrak{s}$. By Theorem 6.7.1, $\geq \mathfrak{s}$ is a good frame with $\lambda_{\mathfrak{s}} = \mu^+$. †

6.8 Uniqueness of Limit Models

Recall that M_α is a (λ, α)-limit model over M_0 iff there is a continuous, increasing chain $\langle M_i \in K_\lambda : i \leq \alpha \rangle$ such that M_{i+1} is universal over M_i for all $i < \alpha$. An easy back-and-forth argument shows that a (λ, θ_1)-limit model and (λ, θ_2)-limit model over M are isomorphic over M if cf $\theta_1 = $ cf θ_2. The general question of uniqueness of limit models asks if this is true for all $\theta_1, \theta_2 < \lambda^+$. Shelah outlines the proof of the uniqueness of limit models from the existence of a good λ-frame, culminating in [Sh:h].II.§4.8. We fill in the details because the outlines Shelah offers are very sparse (see, for instance, [Sh:h].II.§4.11) and to hopefully quell the doubts expressed in [GVV].6. Primarily, we provide a detailed proof of a weakening of [Sh:h].II.§4.11 that constructs a matrix of models, the corner of which is both a (λ, θ_1) and (λ, θ_2) limit model over the same base.

Lemma 6.8.1 (II.§4.11-)**.** *Suppose we have a good λ-frame $\mathfrak{s}$ and*

1. *regular $\theta_1, \theta_2 \leq \lambda$ such that $\delta_1 = \lambda \otimes \theta_1$ and $\delta_2 = \lambda \otimes \theta_2$*

2. *$M \in K_\lambda$.*

Then, we can find functions $\epsilon : \delta_1 \to \delta_2$ and $\eta : \delta_2 \to \delta_1$, an increasing, continuous matrix of models and embeddings $\langle M_{\alpha,\beta} \in K_\lambda : \alpha \leq \delta_1, \beta \leq \delta_2 \rangle$ and coherent $\langle f^{(\alpha_1,\beta_1)}_{(\alpha_0,\beta_0)} : M_{(\alpha_0,\beta_0)} \to M_{(\alpha_1,\beta_1)} \mid \alpha_0 \leq \alpha_1 \leq \delta_1; \beta_0 \leq \beta_1 \leq \delta_2 \rangle$, and $\langle b^1_\alpha \in M_{\alpha+1,\epsilon(\alpha)+1} : \alpha < \delta_1 \rangle$ and $\langle b^2_\beta \in M_{\eta(\beta)+1,\beta+1} : \beta < \delta_2 \rangle$ so

$(\gamma)_1$ *$tp(f^{(\alpha+1,\delta_2)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha)/f^{(\alpha+1,\delta_2)}_{(\alpha,\delta_2)}(M_{\alpha,\delta_2}), M_{\alpha+1,\delta_2})$ does not fork over $f^{(\alpha+1,\delta_2)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1})$.*

$(\gamma)_2$ *$tp(f^{(\delta_1,\beta+1)}_{(\eta(\beta)+1,\beta+1)}(b^2_\beta)/f^{(\delta_1,\beta+1)}_{(\delta_1,\beta)}(M_{\delta_1,\beta}), M_{\delta_1,\beta+1})$ does not fork over $f^{(\delta_1,\beta+1)}_{(\eta(\beta)+1,\beta)}(M_{\eta(\beta)+1,\beta})$.*

$(\delta)_1$ *For all $\alpha < \delta_1, \beta < \delta_2, p \in \mathbf{S}^{bs}(M_{\alpha,\beta+1})$, there are λ many $\alpha' > \alpha$ such that $\beta = \epsilon(\alpha')$ and $tp(b^1_{\alpha'}/f^{(\alpha'+1,\beta+1)}_{(\alpha',\beta+1)}(M_{\alpha',\beta+1}), M_{\alpha'+1,\beta+1})$ is a nonforking extension of $f^{(\alpha'+1,\beta+1)}_{(\alpha,\beta+1)}(p)$.*

$(\delta)_2$ *For all $\alpha < \delta_1, \beta < \delta_2, p \in \mathbf{S}^{bs}(M_{\alpha+1,\beta})$, there are λ many $\beta' > \alpha$ such that $\alpha = \eta(\beta')$ and $tp(b^2_{\beta'}/f^{(\alpha+1,\beta'+1)}_{(\alpha+1,\beta')}(M_{\alpha+1,\beta'}), M_{\alpha+1,\beta'+1})$ is a nonforking extension of $f^{(\alpha+1,\beta'+1)}_{(\alpha+1,\beta)}(p)$.*

The minus indicates that the original lemma has several clauses that aren't needed for this application, so we drop them. Our numbering is, again, to be consistent with [Sh:h]. Here, coherent means that for $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \delta_1$ and $\beta_0 \leq \beta_1 \leq \beta_2 \leq \delta_2$, we have $f^{(\alpha_2,\beta_2)}_{(\alpha_0,\beta_0)} = f^{(\alpha_2,\beta_2)}_{(\alpha_1,\beta_1)} \circ f^{(\alpha_1,\beta_1)}_{(\alpha_0,\beta_0)}$

Proof: There are disjoint $\langle u^1_{\alpha,1} \subset \delta_1 : \alpha < \delta_1, i < \lambda \rangle$ and $\langle u^2_{\beta,i} \subset \delta_2 : \beta < \delta_2, i < \lambda \rangle$ such that, for each $\ell = 1, 2$ and each $\alpha, \gamma < \delta_\ell$ and $i < \lambda$, we have

- $|u^\ell_{\alpha,i}| = \lambda$; and

- $\gamma \in u^\ell_{\alpha,i}$ implies $\gamma > \alpha$.

We want to reindex these sequences based on the types of our matrix models to, for instance, $\langle u^1_{\alpha,\beta,p} \subset \delta_1 : \alpha < \delta_1, \beta < \delta_2, p \in \mathbf{S}^{bs}(M_{\alpha,\beta+1}) \rangle$ by changing the i's to β, p's. Since $|\delta_2| = \lambda$ and K is bs-stable in λ, there is no problem with the cardinalities. However, we have not defined the models $M_{\alpha,\beta}$ yet. Formally, we should index these in terms of α, β, j for $j < \lambda$ and, once $M_{\alpha,\beta+1}$ is defined, enumerate the types. However, this adds more complexity to an already technical proof. Thus, we write them now as $\langle u^1_{\alpha,\beta,p} \subset \delta_1 : \alpha < \delta_1, \beta < \delta_2, p \in \mathbf{S}^{bs}(M_{\alpha,\beta+1}) \rangle$ and $\langle u^2_{\alpha,\beta,p} \subset \delta_2 : \alpha < \delta_1, \beta < \delta_2, p \in \mathbf{S}^{bs}(M_{\alpha+1,\beta}) \rangle$,

noting that they still satisfy the above properties. Define $\epsilon : \delta_1 \to \delta_2$ by $\epsilon(\alpha) = \beta$ iff $\alpha \in u^1_{\alpha_0,\beta,p_0}$ and define $\eta : \delta_2 \to \delta_1$ by $\eta(\beta) = \alpha$ iff $\beta \in u^2_{\alpha,\beta_0,p_0}$. Note that $\epsilon(\alpha) = \beta$ implies $\alpha > \alpha_0$ and $\eta(\beta) = \alpha$ implies $\beta > \beta_0$.

Now we build the rest of our objects by induction so

1. $M_{\alpha,0} = M_{0,\beta} = M$ for all $\alpha \leq \delta_1$ and $\beta \leq \delta_2$.

2. for each $(\alpha, \beta) \in \delta_1 \times \delta_2$,

   a) (i) if $\epsilon(\alpha) < \beta$, $tp(f^{(\alpha+1,\beta)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha) / f^{(\alpha+1,\beta)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha+1,\beta})$ does not fork over $f^{(\alpha+1,\beta)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1})$

   (ii) if $\epsilon(\alpha) = \beta$, then $\alpha \in u^1_{\alpha_0,\beta,p_0}$ for some $\alpha_0 < \alpha$ and $p_0 \in \mathbf{S}^{bs}(M_{\alpha_0,\beta+1})$ and we pick $b^1_\alpha \in M_{\alpha+1,\beta+1}$ that realizes the nonforking extension of $f^{(\alpha+1,\beta+1)}_{(\alpha_0,\beta+1)}(p_0)$ to $f^{(\alpha+1,\beta+1)}_{(\alpha,\beta+1)}(M_{\alpha,\beta+1})$.

   b) (i) if $\eta(\beta) < \alpha$, $tp(f^{(\alpha,\beta+1)}_{(\eta(\beta)+1,\beta+1)}(b^2_\beta) / f^{(\alpha,\beta+1)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha,\beta+1})$ does not fork over $f^{(\alpha,\beta+1)}_{(\eta(\beta)+1,\beta)}(M_{\eta(\beta)+1,\beta})$

   (ii) if $\eta(\beta) = \alpha$, then $\beta \in u^2_{\alpha,\beta_0,p_0}$ for some $\beta_0 < \beta$ and $p_0 \in \mathbf{S}^{bs}(M_{\alpha+1,\beta_0})$ and we pick $b^2_\beta \in M_{\alpha+1,\beta+1}$ that realizes the nonforking extension of $f^{(\alpha+1,\beta+1)}_{(\alpha+1,\beta_0)}(p_0)$ to $f^{(\alpha+1,\beta+1)}_{(\alpha+1,\beta)}(M_{\alpha+1,\beta})$.

Construction: The edges of the matrices are our base cases.

If α or β is limit, then we construct the model via direct unions and check that our conditions hold. So we are in the case where we have $\alpha < \delta_1$ and $\beta < \delta_2$ and we need to construct $M_{\alpha+1,\beta+1}$ and the embeddings given $M_{\alpha+1,\beta}$ and $M_{\alpha,\beta+1}$. Before we construct our model, we do some preparatory work and find $N_\alpha \succ M_{\alpha,\beta+1}$ and $N_\beta \succ M_{\alpha+1,\beta}$; $a_\alpha \in N_\alpha - M_{\alpha,\beta+1}$ and $a_\beta \in N_\beta - M_{\alpha+1,\beta}$; and $n_\alpha \in M_{\alpha+1,\beta}$ so its type over $f^{(\alpha+1,\beta)}_{(\alpha,\beta)}(M_{\alpha,\beta})$ is basic and $n_\beta \in M_{\alpha,\beta+1}$ so its type over $f^{(\alpha,\beta+1)}_{(\alpha,\beta)}(M_{\alpha,\beta})$ is basic.

1. If $\epsilon(\alpha) < \beta$, then we have $tp(f^{(\alpha+1,\beta)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha) / f^{(\alpha+1,\beta)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha+1,\beta})$ is basic, so pick $n_\alpha = f^{(\alpha+1,\beta)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha)$. Otherwise, use the Density to pick n_α arbitrarily. Note that this axiom is not necessary, but helps to make our construction more symmetric.

2. If $\epsilon(\alpha) = \beta$, then $\alpha \in u^1_{\alpha_0,\beta,p_0}$ by construction, so by Extension Existence, there is $tp(a_\alpha / M_{\alpha,\beta+1}, N_\alpha)$ that is a nonforking extension of $f^{(\alpha,\beta+1)}_{(\alpha_0,\beta+1)}(p_0)$. Otherwise, pick them arbitrarily. Note that $\alpha_0 < \alpha$, so $M_{\alpha_0,\beta+1}$ has been constructed prior to this step, so this enumeration is well defined.

3. If $\eta(\beta) < \alpha$, then we have $tp(f^{(\alpha,\beta+1)}_{(\eta(\beta)+1,\beta+1)}(b^2_\beta) / f^{(\alpha,\beta+1)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha,\beta+1})$ is basic, so pick $n_\beta = f^{(\alpha,\beta+1)}_{(\eta(\beta)+1,\beta+1)}(b^2_\beta)$. Otherwise, pick n_β arbitrarily.

4. If $\eta(\beta) = \alpha$, then $\beta \in u^2_{\alpha,\beta_0,p_0}$, so find, by Extension Existence, $tp(a_\beta / M_{\alpha+1,\beta}, N_\beta)$ that is a nonforking extension of $f^{(\alpha+1,\beta)}_{(\alpha+1,\beta_0)}(p_0)$. Otherwise, pick them arbitrarily. As above, $\beta_0 < \beta$, so this is well defined.

Now that we have this, we apply Non-Forking Amalgamation to get the following

$$
\begin{array}{ccc}
N_\beta & \xrightarrow{\quad g_\beta \quad} & M_{\alpha+1,\beta+1} \\
\uparrow & & \uparrow \\
M_{\alpha+1,\beta} & & g_\alpha \\
\uparrow {\scriptstyle f^{(\alpha+1,\beta)}_{(\alpha,\beta)}} & & \\
M_{\alpha,\beta} \xrightarrow[f^{(\alpha,\beta+1)}_{(\alpha,\beta)}]{} M_{\alpha,\beta+1} & \longrightarrow & N_\alpha
\end{array}
$$

Set $f^{(\alpha+1,\beta+1)}_{(\alpha+1,\beta)} = g_\beta \upharpoonright M_{\alpha+1,\beta}$ and $f^{(\alpha+1,\beta+1)}_{(\alpha,\beta+1)} = g_\alpha \upharpoonright M_{\alpha,\beta+1}$. Then compose the rest of the embeddings to make everything coherent.

1. If $\epsilon(\alpha) < \beta$, then $\epsilon(\alpha) < \beta + 1$ and nonforking amalgamation tells us (after a little rewriting) that

$$
tp(f^{(\alpha+1,\beta+1)}_{(\alpha+1,\epsilon(\alpha)+1)}(b_\alpha + 1)/f^{(\alpha+1,\beta+1)}_{(\alpha,\beta+1)}(M_{\alpha,\beta+1}), M_{\alpha+1,\beta+1})
$$
$$
\text{does not fork over } f^{(\alpha+1,\beta+1)}_{(\alpha,\beta)}(M_{\alpha,\beta}) \tag{6.1}
$$

By induction, we have that $tp(f^{(\alpha+1,\beta)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha)/f^{(\alpha+1,\beta)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha+1,\beta})$ does not fork over $f^{(\alpha+1,\beta)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1})$. Applying $f^{(\alpha+1,\beta+1)}_{(\alpha+1,\beta)}$ to this and applying Monotonicity, we get that

$$
tp(f^{(\alpha+1,\beta+1)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha)/f^{(\alpha+1,\beta+1)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha+1,\beta+1})
$$
$$
\text{does not fork over } f^{(\alpha+1,\beta+1)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1}) \tag{6.2}
$$

Then, we apply Transitivity to Eqs. (9.1) and (9.2) and get that
$tp(f^{(\alpha+1,\beta+1)}_{(\alpha+1,\epsilon(\alpha)+1)}(b_\alpha+1)/\, f^{(\alpha+1,\beta+1)}_{(\alpha,\beta+1)}(M_{\alpha,\beta+1}), M_{\alpha+1,\beta+1})$ does not fork over $f^{(\alpha+1,\beta+1)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1})$, as desired.

2. If $\epsilon(\alpha) = \beta$, then we set $b^1_\alpha = g_\alpha(a_\alpha) \in M_{\alpha+1,\beta+1}$. We know that $a_\alpha \vDash f^{(\alpha,\beta+1)}_{(\alpha_0,\beta+1)}(p_0)$, so b^1_α realizes $f^{(\alpha+1,\beta+1)}_{(\alpha_0,\beta+1)}(p_0)$. Additionally, we picked a_α so $tp(a_\alpha/M_{\alpha,\beta+1}, N_\alpha)$ does not fork over $f^{(\alpha,\beta+1)}_{(\alpha_0,\beta+1)}(M_{\alpha,\beta+1})$. Applying $g_\alpha \supset f^{(\alpha+1,\beta+1)}_{(\alpha,\beta+1)}$ to this and using Monotonicity, we get that $tp(b^1_\alpha/f^{(\alpha+1,\beta+1)}_{(\alpha,\beta+1)}(M_{\alpha,\beta+1}), M_{\alpha+1,\beta+1})$ does not fork over $f^{(\alpha+1,\beta+1)}_{(\alpha_0,\beta+1)}(M_{\alpha_0,\beta+1})$.

3. If $\eta(\beta) < \alpha$ or $\eta(\beta) = \alpha$, the proof is symmetric, since our goal and our set-up is symmetric.

This is enough: Now we want to show that our construction has fulfilled the lemma.

$(\gamma)_1$ Set $\alpha < \delta_1$. For each $\beta > \epsilon(\alpha)$, we know that $tp(f^{(\alpha+1,\beta)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha)/f^{(\alpha+1,\beta)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha+1,\beta})$ does not fork over $f^{(\alpha+1,\beta)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1})$ by 2.(a)(i) of the construction. If we apply the map $f^{(\alpha+1,\delta_2)}_{(\alpha+1,\beta)}$ and use Monotonicity, we get that $tp(f^{(\alpha+1,\delta_2)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha)/f^{(\alpha+1,\delta_2)}_{(\alpha,\beta)}(M_{\alpha,\beta}), M_{\alpha+1,\delta_2})$ does not fork over $f^{(\alpha+1,\delta_2)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1})$ for every $\epsilon(\alpha) < \beta < \delta_2$. Then, by Continuity, we have that $tp(f^{(\alpha+1,\delta_2)}_{(\alpha+1,\epsilon(\alpha)+1)}(b^1_\alpha)/\, f^{(\alpha+1,\delta_2)}_{(\alpha,\delta_2)}(M_{\alpha,\delta_2}), M_{\alpha+1,\delta_2})$ does not fork over $f^{(\alpha+1,\delta_2)}_{(\alpha,\epsilon(\alpha)+1)}(M_{\alpha,\epsilon(\alpha)+1})$, as desired.

$(\delta)_1$ Fix $\alpha < \delta_1, \beta < \delta_2, p \in \mathbf{S}^{bs}(M_{\alpha,\beta+1})$. Then $u^1_{\alpha,\beta,p} = \epsilon^{-1}(\{\beta\})$ has size λ and, for every such α', $tp(b^1_{\alpha'}/f^{(\alpha'+1,\beta+1)}_{(\alpha',\beta+1)}(M_{\alpha',\beta+1}), M_{\alpha'+1,\beta+1})$ is a nonforking extension of $f^{(\alpha'+1,\beta+1)}_{(\alpha,\beta+1)}(p)$ by 2.(a).(oo).

$(\gamma)_2, (\delta)_2$ Similarly.

This completes the proof of the lemma. †

For reference and, in particular, for use in the previous chapter, we note that the only frame properties used were Amalgamation, Density, bs-stability, Monotonicity, Transitivity, Symmetry, Extension Existence, and Continuity. In particular, Continuity was only used for chains of length θ_1 and θ_2. We can now prove the uniqueness of limit models.

Theorem 6.8.2 ([Sh:h]II.§4.8). *If we have a good λ-frame, then K_λ has unique limit models.*

Proof: Let N_1 be a (λ, θ_1)-limit model over M and N_2 be a (λ, θ_2)-limit model over M. Apply the lemma above to get functions $\epsilon : \delta_1 \to \delta_2$ and $\eta : \delta_2 \to \delta_1$ and an increasing, continuous matrix of models and embeddings $\langle M_{\alpha,\beta} \in K_\lambda : \alpha \le \delta_1, \beta \le \delta_2 \rangle$ and coherent $\langle f^{(\alpha_1,\beta_1)}_{(\alpha_0,\beta_0)} : M_{(\alpha_0,\beta_0)} \to M_{(\alpha_1,\beta_1)} \mid \alpha_0 \le \alpha_1 \le \delta_1; \beta_0 \le \beta_1 \le \delta_2 \rangle$ and $\langle b^1_\alpha \in M_{\alpha+1,\epsilon(\alpha)+1} : \alpha < \delta_1 \rangle$ and $\langle b^2_\beta \in M_{\eta(\beta)+1,\beta+1} : \beta < \delta_2 \rangle$ as there. By renaming, we get increasing continuous $\langle M^{\delta_2}_\alpha : \alpha \le \delta_1 \rangle$ and $\langle M^{\delta_1}_\beta : \beta \le \delta_2 \rangle$ such that $M^{\delta_2}_0 = M^{\delta_1}_0 = M$ and $M^{\delta_1}_{\delta_2} = M^{\delta_2}_{\delta_1}$, which is the renaming of M_{δ_1,δ_2} with the property

$(*)_1$ if $\alpha < \delta_1$ and $p \in \mathbf{S}^{bs}(M^{\delta_2}_\alpha)$, then there are λ-many $\alpha' > \alpha$ such that $tp(b^1_{\alpha'}/M^{\delta_2}_{\alpha'}, M^{\delta_2}_{\alpha'+1})$ is a nonforking extension of p.

$(*)_2$ if $\beta < \delta_2$ and $p \in \mathbf{S}^{bs}(M^{\delta_1}_\beta)$, there there are λ-many $\beta' > \beta$ such that $tp(b^2_{\beta'}/M^{\delta_2}_{\beta'}, M^{\delta_2}_{\beta'+1})$ is a nonforking extension of p.

Once we have established these, we use [Sh:h].II.§4.3 (see Theorem 6.8.3) to see that $M^{\delta_2}_{\delta_1}$ is (λ, θ_1)-limit over M and $M^{\delta_1}_{\delta_2}$ is (λ, θ_2)-limit over M. Then, by uniqueness of limit models of the same length, we get that

$$N_1 \cong_M M^{\delta_1}_{\delta_1} = M^{\delta_1}_{\delta_2} \cong_M N_2$$

†

For reference, [Sh:h].II.§.4.3 is stated below and has a detailed proof at the reference and uses only Density and Local Character.

Theorem 6.8.3 (Shelah). *Assume $\mathfrak{s}$ is a good λ-frame and*

1. *$\delta < \lambda^+$ is a limit ordinal divisible by λ;*

2. *$\langle M_\alpha \in K_\lambda : \alpha \le \delta \rangle$ is increasing and continuous; and*

3. *if $i < \delta$ and $p \in \mathbf{S}^{bs}_{\mathfrak{s}}(M_i)$, then for λ-many ordinals $j \in (i, \delta)$, there is $c \in M_{j+1}$ realizing the nonforking extension of p in $\mathbf{S}^{bs}(M_j)$.*

Then M_δ is $(\lambda, cf\,\delta)$-limit over M_0 and (therefore) universal over it.

6.9 Good Frames in Hart-Shelah

In this section, we show that some additional hypothesis is necessary to extend a good λ-frame $\mathfrak{s}$ to a good frame $\geq \mathfrak{s}$. This example was included in response to a referee question about Theorem 6.1.3, and I would like to thank the referee for the question and Alexei Kolesnikov for helpful discussions.

We recall the main result from [BK09].

Theorem 6.9.1 ([BK09]). *For each $n < \omega$, there is $\phi_n \in L_{\omega_1,\omega}$ so*

1. *ϕ_n is categorical in all $\mu \leq \aleph_n$;*

2. *ϕ_n is not $\aleph_n$-stable;*

3. *ϕ_n is not categorical in any $\mu > \aleph_n$;*

4. *ϕ_n has the disjoint amalgamation property; and*

5. *if $n > 0$, then*

   a) *ϕ_n is $(\aleph_0, \aleph_{n-1})$-tame; in fact, Galois types over models of size $\leq \aleph_{n-1}$ are first order, syntactic types;*

   b) *ϕ_n is μ-stable for $\mu < \aleph_n$; and*

   c) *ϕ_n is not $(\aleph_{n-1}, \aleph_n)$-tame.*

Note that the sentences ϕ_n have been reindexed (as compared to [BK09]) in order to avoid unnecessary subscripts such that "ϕ_n" here is "ϕ_{n+2}" there. We will not give the full definition of ϕ_n (it can be found in [BK09].§1), but will outline some of the key features. Each model M consists of an index set $I(M)$ (often called the spine) and additional elements built off of this spine, mainly variously indexed copies of $\mathbb{Z}_2$ including fibers over $[I(M)]^{n+2}$ consisting of elements of from the direct sum of $\mathbb{Z}_2$ indexed by $[I(M)]^{n+2}$. Included in the language are also various projection functions and addition functions. Added to this is an $(n+3)$-ary predicate Q which codes the addition of $n+2$ many fibers without explicitly including it.

[BK09] improves on (and introduces a minor correction to) the original analysis in [HaSh323]. In addition to the theorem above, they show that the class of models of ϕ_n is model complete ([BK09].4.8).

If $n > 0$, then ϕ_n is categorical in at least two successive cardinals ($\aleph_n$ and $\aleph_{n-1}$, for instance), so the results of Shelah [Sh576] imply that there is a good λ-frame under favorable cardinal arithmetic (recall Theorem 2.3.7). However, the Hart-Shelah example is well-enough understood that cardinal arithmetic is not needed for the existence of a good λ-frame in this case. Additionally, we have the existence of a good $\aleph_0$-frame in ϕ_0, which is only categorical in $\aleph_0$, a result not predicted by [Sh:h].II.

Theorem 6.9.2. *Fix $n < \omega$ and $\mu < \aleph_n$. There is $\mathfrak{s}^n_\mu$ such that*

1. *$\mathfrak{s}^n_\mu$ is a good μ-frame for ϕ_n;*

2. *if $\mu < \mu' < \aleph_n$, then $(\geq \mathfrak{s}^n_\mu) \restriction \mu' = \mathfrak{s}^n_{\mu'}$; and*

3. *if $\mu' \geq \aleph_n$, then $(\geq \mathfrak{s}^n_\mu) \restriction \mu'$ is a good μ-frame for ϕ_n except for Uniqueness and Basic Stability, both of which fail.*

Although this proof does not assume any cardinal arithmetic and, therefore, does not use the results of [Sh576] to find a frame, the frame definition given is inspired by that frame.

Proof: Fix $n < \omega$. Then, for this proof, we set K^n to be the models of ϕ_n from [BK09] and set $M \prec^n N$ iff $M \prec_{L_{\omega_1,\omega}} N$. This is the same as $M \subset N$ by model completeness.

Fix $\mu < \aleph_n$. We define the frame $\mathfrak{s}_\mu^n = (K_\mu^n, \underset{\mu}{\overset{n}{\downarrow}}, \mathbf{S}_{\mu,n}^{bs})$ by:

- for $M \in K^n$, $tp(a/M, N) \in \mathbf{S}_{\mu,n}^{bs}(M)$ iff $a \in I(M) - I(N)$; and

- $\underset{\mu}{\overset{n}{\downarrow}}(M_0, M_1, a, M_3)$ iff $M_0 \prec^n M_1 \prec^n M_3$ and $a \in I(M_3) - I(M_1)$.

From the definitions, it then follows that, for any $M \in K_{\geq\mu}$,

- $tp(a/M, N) \in \mathbf{S}_{\geq(\mathfrak{s}_\mu^n)}^{bs}(M)$ iff $a \in I(M) - I(N)$; and

- $\underset{\geq(\mathfrak{s}_\mu^n)}{\downarrow}(M_0, M_1, a, M_3)$ iff $M_0 \prec^n M_1 \prec^n M_3$ and $a \in I(M_3) - I(M_1)$.

This establishes 2. To show 1. and 3., we will show that $\geq (s_\mu^n)$ satisfies all of the good frame axioms except bs-Stability and Uniqueness and that bs-Stability and Uniqueness hold if the models are of size $< \aleph_n$. We do this by going through the axioms of Definition 2.3.5 and showing that they hold. For notational ease, set $K := K^n$, $\mathfrak{s} := \mathfrak{s}_\mu^n$, $\downarrow = \underset{\mu}{\overset{n}{\downarrow}}$, and $\mathbf{S}^{bs} := \mathbf{S}_{\mu,n}^{bs}$. Many of the frame properties follow immediately from the definition and the observation that, given $p \in \mathbf{S}^{bs}(M)$ and $M_0 \prec M$, p does not fork over M_0. The non-trivial arguments are given below.

(C) By [BK09].3.1, K has the stronger property of disjoint amalgamation. By [BK09].2.15, K is categorical in $\aleph_0$. Combining this with amalgamation implies that K has joint embedding. We know that K has arbitrarily large models by [BK09].1.3. This, plus amalgamation and joint embedding from above, show K has no maximal models; see [BLS1003].3.3. Thus, K_μ has no maximal models. This can also be seen directly be extending the spine, I.

(D) (c) **Density:** The elements of M are determined (up to isomorphism) by $I(M)$. Thus, $M \not\succeq N$ implies $I(M) \subsetneq I(N)$.

(d) bs-**stability:** Below $\aleph_n$, full stability holds by [BK09].7.1; this clearly implies bs-stability. At $\aleph_n$ and above, the proof of [BK09].6.1 show that there are the maximal number of Galois types of elements from I.

(E) (e) **Uniqueness:** By [BK09].5.1 , Galois types of finite tuples over models of size less than $\aleph_n$ are syntactic, first-order types. Any two non-algebraic elements in the spine have the same syntactic type, so Uniqueness holds. At $\aleph_n$ and above, the proof of [BK09].6.8 shows that tameness for basic types fails, so, by Theorem 6.2.1, Uniqueness fails as well.

(f) **Symmetry:** Let $M_0 \prec M_1 \prec M_3$ with $a_1 \in I(M_1) - I(M_0)$ and $a_2 \in I(M_3) - I(M_1)$. Take M_2 to be the substructure generated by M_0 and a_2 in M_3. Then $I(M_2) = I(M_0) \cup \{a_2\}$ and, in particular, $a_1 \notin I(M_2)$, as desired.

(g) **Extension Existence:** Let M and $p \in \mathbf{S}_{\geq\mathfrak{s}}^{bs}(M)$ and $N \succ M$. Set $p = tp(a/M, N')$ and find a disjoint amalgam $N^* \succ N$ and $f : N' \to_M N^*$. Then $q = tp(f(a)/N, N^*)$ is a nonforking extension of p. $\dagger$

In addition to showing that some additional hypothesis is needed to extend a good frame, this example gives a non-trivial example of a frame in ZFC, i.e. without cardinal arithmetic assumptions. Additionally, this gives an example of a partially categorical AEC with a supersimple-like independence notion, that is, one that has Local Character, Extension Existence, etc., but not Uniqueness.

A Representation Theorem for Continuous Logic

7.1 Introduction

In the spirit of Chang and Shelah's presentation results, this chapter gives a presentation theorem for continuous logic. That is, given a continuous signature L, we find a discrete signature L^+ and an $L^+_{\omega_1,\omega}$-theory T_{dense} such that the continuous L-structures correspond to the L^+-structures that model T_{dense}. Obviously, this correspondence cannot be exact as continuous structures are complete and this property is not $L_{\omega_1,\omega}$ axiomatizable. Conversely, T_{dense} will have models in cardinalities of countable cofinality, which cannot occur for most metrics.

Instead, we avoid the question of topological completeness by focusing on the dense subsets of complete models. Dense sets are not quite restrictive enough, so we introduce nicely dense sets to require them to be closed under functions.

Definition 7.1.1. *Given a continuous model M and a set $A \subset |M|$, we say that A is* nicely dense *iff A is dense in the metric structure $(|M|, d^M)$ and A is closed under the functions of M.*

7.2 Models and Theories

In the what follows, we will often want to prove similar results for both "greater than" and "less than." In order to avoid writing everything twice, we often use $\square$ to stand in for both $\geq$ and $\leq$. Thus, asserting a statement for "$r \square s$" means that that statement is true for "$r \geq s$" and for "$r \leq s$."

Our goal is to translate the functional formulas of L into classic, true/false formulas. We do this by encoding relations into L^+ that are intended to specify the value of ϕ by deciding if it is above or below each possible value. To ensure that the size of the language doesn't grow, we take advantage of the separability or $\mathbb{R}$ and only compare each ϕ to the rationals in $[0, 1]$. For notational ease, we set $\mathbb{Q}' := [0, 1] \cap \mathbb{Q}$.

The main thesis of this chapter is that model-theoretic properties of continuous first order structures can be translated to model-theoretic (but typically quantifier free) properties of discrete structures that model a specific theory in an expanded language. The main theorem about this presentation is the following:

Theorem 7.2.1. *Let L be a continuous language. Then there is*

(a) *a discrete language L^+;*

(b) *an $L^+_{\omega_1,\omega}$ theory T_{dense};*

(c) *a map from continuous L-structures M and nicely dense subsets A to discrete L^+-structures M_A that model T_{dense};*

(d) *a map from discrete L^+ structures A that model T_{dense} to continuous L-structures $\overline{A}$*

with the properties that

1. *$M_A \vDash A$ has universe A and, for any $\mathbf{a} \in A$, $\phi(\mathbf{x}) \in Fml^c L$, $r \in \mathbb{Q}'$, and $\square$ standing for $\geq$ and $\leq$, we have*

$$M_A \vDash R_{\phi \square r}[\mathbf{a}] \iff \phi^M(\mathbf{a}) \square r$$

2. *A is a dense subset of $\overline{A}$ and, for any $\mathbf{a} \in A$, $\phi(\mathbf{x}) \in Fml^c L$, $r \in \mathbb{Q}'$, and $\square$ standing for $\geq$ and $\leq$, we have*

$$A \vDash R_{\phi \square r}[\mathbf{a}] \iff \phi^{\overline{A}}(\mathbf{a}) \square r$$

3. *these maps are (essentially) each other's inverse. That is, given any nicely dense $A \subset M$, we have $M \cong_A \overline{M_A}$ and, given any L^+-structure $A \vDash T_{dense}$, we have $(\overline{A})_A = A$.*

The "essentially" in the last clause comes from the fact that completions are not technically unique as the objects selected as limits can vary, but this fairly pedantic point is the only obstacle.

Proof: Our proof is long, but straightforward. First, we will define L^+ and T_{dense}. Then, we will introduce the map $(M, A) \mapsto M_A$ and prove it satisfies (1). After this, we will introduce the other map $A \mapsto \overline{A}$ and prove (2). Finally, we will prove that they satisfy (3).

Defining the new language and theory

We define the language L^+ to be

$$\langle F_i^+, R_{\phi(\mathbf{x}) \geq r}, R_{\phi(\mathbf{x}) \leq r} \rangle_{i < L_F, \phi(\mathbf{x}) \in Fml^c(L), r \in \mathbb{Q}'}$$

with the arity of F_i^+ matching the arity of F_i and the arity of $R_{\phi(\mathbf{x}) \square r}$ matching $\ell(\mathbf{x})$. Since we only use a full (dense) set of connectives, we have ensured that $|L^+| = |L| + \aleph_0$.

We define $T_{dense} \subset L^+_{\omega_1, \omega}$ to be the universal closure of all of the following formulas ranging over all continuous formulas $\phi(\mathbf{z})$ and $\psi(\mathbf{z}')$, all terms $\tau(z, \mathbf{z}'')$, and all $r, s \in \mathbb{Q}'$ and $t \in \mathbb{Q}' - \{0\}$. We have divided them into headings so that their meaning is (hopefully) more clear. When we refer to specific sentences of T_{dense} later, we reference the ordering in this list. As always, a $\square$ in a formula means that it should be included with both a '$\geq$' and a '$\leq$' replacing the $\square$.

1. **The ordered structure of $\mathbb{R}$**

   a) $\neg R_{\phi(\mathbf{z}) \geq r}(\mathbf{x}) \implies R_{\phi(\mathbf{z}) \leq r}(\mathbf{x})$

   b) $\neg R_{\phi(\mathbf{z}) \leq r}(\mathbf{x}) \implies R_{\phi(\mathbf{z}) \geq r}(\mathbf{x})$

   c) If $r > s$, then include $\neg R_{\phi(\mathbf{z}) \geq r}(\mathbf{x}) \vee \neg R_{\phi(\mathbf{z}) \leq s}(\mathbf{x})$

   d) If $r \geq s$, then include

   - $R_{\phi(\mathbf{z}) \leq s}(\mathbf{x}) \implies R_{\phi(\mathbf{z}) \leq r}(\mathbf{x})$; and
   - $R_{\phi(\mathbf{z}) \geq r}(\mathbf{x}) \implies R_{\phi(\mathbf{z}) \geq s}(\mathbf{x})$

   e) $R_{\phi(\mathbf{z}) \geq r}(\mathbf{x}) \vee R_{\phi(\mathbf{z}) \leq r}(\mathbf{x})$

   f) $\wedge_{n < \omega} \vee_{r, s \in \mathbb{Q}', |r-s| < \frac{1}{n}} R_{\phi \leq r}(\mathbf{x}) \wedge R_{\phi \geq s}(\mathbf{x})$

   g) $(\wedge_{n < \omega} R_{\phi(\mathbf{z}) \geq r - \frac{1}{n}}(\mathbf{x})) \implies R_{\phi(\mathbf{z}) \geq r}(\mathbf{x})$

   h) $(\wedge_{n < \omega} R_{\phi(\mathbf{z}) \leq r + \frac{1}{n}}(\mathbf{x})) \implies R_{\phi(\mathbf{z}) \leq r}(\mathbf{x})$

2. **Construction of formulas**

   a) $R_{\phi(\mathbf{z}) \geq 0}(\mathbf{x}) \wedge R_{\phi(\mathbf{z}) \leq 1}(\mathbf{x})$

b) $\neg R_{0 \geq t}(\mathbf{x}) \wedge \neg R_{1 \leq 1-t}(\mathbf{x})$;

c) $R_{\frac{\phi(\mathbf{z})}{2} \geq r}(\mathbf{x}) \iff R_{\phi(\mathbf{z}) \geq 2r}(\mathbf{x})$

d) $R_{\frac{\phi(\mathbf{z})}{2} \leq r}(\mathbf{x}) \iff R_{\phi(\mathbf{z}) \leq 2r}(\mathbf{x})$;

e) $R_{\phi(\mathbf{z}) \dot{-} \psi(\mathbf{z}') \geq r}(\mathbf{x}, \mathbf{x}') \iff \vee_{s \in \mathbb{Q}'}(R_{\psi(\mathbf{z}') \leq s}(\mathbf{x}') \wedge R_{\phi(\mathbf{z}) \geq r+s}(\mathbf{x}))$

f) $R_{\phi(\mathbf{z}) \dot{-} \psi(\mathbf{z}') \leq r}(\mathbf{x}, \mathbf{x}') \iff \vee_{s \in \mathbb{Q}'}(\neg R_{\psi(\mathbf{z}') \leq s}(\mathbf{x}') \wedge R_{\phi(\mathbf{z}) \leq r+s}(\mathbf{x}))$;

g) $R_{\sup_y \tau(y,\mathbf{y}) \leq r}(\mathbf{x}) \iff \forall x R_{\tau(y,\mathbf{y}) \leq r}(x, \mathbf{x})$

h) $R_{\sup_y \tau(y,\mathbf{y}) \geq r}(\mathbf{x}) \iff \wedge_{n<\omega} \exists x R_{\tau(y,\mathbf{y}) \geq r - \frac{1}{n}}(x, \mathbf{x})$;

i) $R_{\inf_y \tau(y,\mathbf{y}) \leq r}(\mathbf{x}) \iff \wedge_{n<\omega} \exists x R_{\tau(y,\mathbf{y}) \leq r + \frac{1}{n}}(x, \mathbf{x})$

j) $R_{\inf_y \tau(y,\mathbf{y}) \geq r}(\mathbf{x}) \iff \forall x R_{\phi(y,\mathbf{y}) \geq r}(x, \mathbf{x})$;

k) $R_{\phi(y,\mathbf{y}) \square r}(\tau(\mathbf{x}'), \mathbf{x}) \iff R_{\phi(\tau(\mathbf{y}'),\mathbf{x}) \square r}(\mathbf{x}', \mathbf{x})$

3. **Metric structure**

   a) $R_{d(y,y') \leq 0}(x, x') \iff x = x'$;

   b) $R_{d(y,y') \square r}(x, x') \iff R_{d(y,y') \square r}(x', x)$;

   c) $\wedge_{r \in \mathbb{Q}'}(R_{d(y,y') \geq r}(x, x') \implies \forall x'' \vee_{s \in \mathbb{Q}' \cap [0,r]} R_{d(y,y') \geq s}(x, x'') \wedge R_{d(y,y') \geq r-s}(x'', x'))$

4. **Uniform Continuity**

   a) For each $r, s \in \mathbb{Q}'$ and $i < L_F$ such that $s < \Delta_{F_i}(r)$, we include the sentence

   $$\wedge_{i<n} R_{d(z,z') \leq s}(x_i, y_i) \implies R_{d(z,z') \leq r}(F_i(\mathbf{x}), F_i(\mathbf{y}))$$

   b) For each $r, s \in \mathbb{Q}'$ and $j < L_R$ such that $s < \Delta_{R_j}(r)$, we include the sentence

   $$\wedge_{i<n} R_{d(z,z') \leq s}(x_i, y_i) \implies (R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}') \leq r}(\mathbf{x}, \mathbf{y}) \wedge R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}') \leq r}(\mathbf{y}, \mathbf{x}))$$

We have been careful about the specific enumeration of these axioms for a reason. If the original continuous language is countable, then T_{dense} is countable. In particular, we could take the conjunction of it and make it a single $L^+_{\omega_1,\omega}$ sentence. This means that it is expressible in a countable fragment of $L_{\omega_1,\omega}$. Countable fragments are the most well-studied infinitary languages and many of the results in, say, Keisler [Kei71] use these fragments. In general, T_{dense} is expressible in a $|L| + \aleph_0$ sized fragment of $L^+_{\omega_1,\omega}$.

From continuous to discrete...

This is the easier of the directions. We define the structure M_A so that all of the "intended" correspondences hold and everything works out well.

Suppose we have a continuous L-structure M and a nicely dense subset A. Now we define an L^+ structure M_A by

1. the universe of M_A is A;

2. $(F_i^+)^{M_A} = F_i^M \upharpoonright A$ for $i < L_F$; and

3. for $r \in \mathbb{Q}'$ and $\phi(\mathbf{x}) \in Fml^c(L)$, set

$$R^{M_A}_{\phi \square r} = \{\mathbf{a} \in A : \phi^M(\mathbf{a}) \square r\}$$

This is an L^+-structure since it is closed under functions. The real meat of this part is the following claim, which is (1) from the theorem.

Claim: $M_A \vDash T_{dense}$ and, for any $\mathbf{a} \in A$, $\phi(\mathbf{x}) \in Fml^c L$, $r \in \mathbb{Q}'$, and $\square = \geq, \leq$, we have

$$M_A \vDash R_{\phi \square r}[\mathbf{a}] \iff \phi^M(\mathbf{a}) \square r$$

Proof of Claim: This is all straightforward. From the definition, we know that, for any $\mathbf{a} \in A$ and formula $\phi(\mathbf{x}) \in Fml^c(L)$ and $\square \in \{\geq, \leq\}$, we have

$$M_A \vDash R_{\phi \square r}[\mathbf{a}] \iff \phi^M(\mathbf{a}) \square r$$

This gives an easy proof of the fact that $M_A \vDash T_{dense}$ because they are all just true facts if '$R_{\phi \square r}(\mathbf{a})$' is replaced by '$\phi \square r$.' $\dagger_{Claim}$

...and back again

This is the harder direction. We want to 'read out' the L-structure that A is a dense subset of from the L^+ structure. First, we use the axioms of T_{dense} to show that we can read out the metric and relations of L from the relations of L^+ and that these are well-defined. Then we complete A and use the uniform continuity of the derived relations to expand them to the whole structure. In the first direction, T_{dense} could have been any collection of true sentences about continuous structures and the real line, but this direction makes it clear that the axioms chosen are necessary.

Suppose that we have an L^+-structure A that models T_{dense}. The following claim is an important step in reading out the relations of the completion of A from A.

Claim 7.2.2. *For any $\phi(\mathbf{x}) \in Fml^c(L)$ and $\mathbf{a} \in A$, we have*

$$\sup\{t \in \mathbb{Q}' : A \vDash R_{\phi(\mathbf{x}) \leq t}(\mathbf{a})\} = \inf\{t \in \mathbb{Q}' : A \vDash R_{\phi(\mathbf{x}) \geq t}(\mathbf{a})\}$$

Proof: We show this equality by showing two inequalities.

- Let $r \in \{t \in \mathbb{Q}' : A \vDash R_{\phi(\mathbf{x}) \leq t}(\mathbf{a})\}$ and $s \in \{t \in \mathbb{Q}' : A \vDash R_{\phi(\mathbf{x}) \geq t}(\mathbf{a})\}$. Then

$$A \vDash R_{\phi(\mathbf{x}) \geq r}(\mathbf{a}) \wedge R_{\phi(\mathbf{x}) \leq s}(\mathbf{a})$$

Then, since M^+ satisfies 1c, we must have $r \leq s$. Thus $\sup\{t \in \mathbb{Q}' : A \vDash R_{\phi(\mathbf{x}) \leq t}(\mathbf{a})\} \leq \inf\{t \in \mathbb{Q}' : A \vDash R_{\phi(\mathbf{x}) \geq t}(\mathbf{a})\}$.

- By 1f, we have

$$A \vDash \wedge_{n < \omega} \vee_{r,s \in \mathbb{Q}';|r-s|<\frac{1}{n}} R_{\phi(\mathbf{x}) \leq r}(\mathbf{a}) \wedge R_{\phi(\mathbf{x}) \geq s}(\mathbf{a})$$

Let $\epsilon > 0$. Then there is $n_0 < \omega$ such that $\epsilon > \frac{1}{n_0}$. By the above, there are $r, s \in \mathbb{Q}'$ such that $|r - s| < \frac{1}{n_0}$ and

$$M^+ \vDash R_{\phi(\mathbf{x}) \leq r}(\mathbf{a}) \wedge R_{\phi(\mathbf{x}) \geq s}(\mathbf{a})$$

As above, 1c implies $r \geq s$, so we have $r - s < \frac{1}{n_0} < \epsilon$. Thus $r < s + \epsilon$ and $s \in \{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x}) \leq t}(\mathbf{a})\}$ and $r \in \{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x}) \geq t}(\mathbf{a})\}$. Then, $\inf\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x}) \geq t}(\mathbf{a})\} \leq \sup\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x}) \leq t}(\mathbf{a})\}$.
$$\dagger_{Claim}$$

The first relation that we need is the metric. Given $a, b \in A$, we set

$$
\begin{aligned}
D(a, b) \quad &:= \quad \sup\{r \in \mathbb{Q}' : A \models R_{d(x,y) \geq r}[a, b]\} \\
&= \quad \inf\{r \in \mathbb{Q}' : A \models R_{d(x,y) \leq r}[a, b]\}
\end{aligned}
$$

These definitions are equivalent by Claim 7.2.2. We show that this is indeed a metric on A.

Claim 7.2.3. $(|A|, D)$ *is a metric space.*

Proof: We go through the metric space axioms. Let $a, b \in |A|$.

1.

$$
\begin{aligned}
D(a, b) = 0 \quad &\Longrightarrow \quad \inf\{r \in \mathbb{Q}' : A \models R_{d(x,y) \leq r}(a, b)\} = 0 \\
&\Longrightarrow \quad \forall n < \omega \exists r_n \in \mathbb{Q}' \text{ so } A \models R_{d(x,y) \leq r_n}(a, b) \text{ and } r_n \leq \frac{1}{n} \\
&\Longrightarrow_{1d} \quad \forall n < \omega, A \models R_{d(x,y) \leq \frac{1}{n}}(a, b) \\
&\Longrightarrow_{1h} \quad A \models R_{d(x,y) \leq 0}(a, b) \\
&\Longrightarrow \quad a = b
\end{aligned}
$$

$$
\begin{aligned}
a = b \quad &\Longrightarrow \quad A \models R_{d(x,y) \leq 0}(a, b) \\
&\Longrightarrow \quad \inf\{r \in \mathbb{Q}' : A \models R_{d(x,y) \leq r}(a, b)\} = 0 \\
&\Longrightarrow \quad D(a, b) = 0
\end{aligned}
$$

2.

$$D(a, b) = \sup\{r \in \mathbb{Q}' : A \models R_{d(x,y) \geq r}(a, b)\} =_{3b} \sup\{r \in \mathbb{Q}' : A \models R_{d(x,y) \geq r}(b, a)\} = D(b, a)$$

3. Let $c \in |A|$. We want to show $D(a, c) \leq D(a, b) + D(b, c)$. It is enough to show

$$\forall r \in \mathbb{Q}'(D(a, c) \geq r \implies D(a, b) + D(b, c) \geq r)$$

Thus, let $r \in \mathbb{Q}'$ and suppose $D(a, c) \geq r$. Then $\sup\{s \in \mathbb{Q}' : A \models R_{d(x,y) \geq s}(a, c)\} \geq r$. By 3c, this means

$$\sup\{s \in \mathbb{Q}' : A \models \vee_{t \in \mathbb{Q}' \cap [0,s]} R_{d(x,y) \geq t}(a, b) \wedge R_{d(x,y) \geq s-t}(b, c)\} \geq r$$

Fix $n < \omega$. There is some $s_n \in \mathbb{Q}'$ such that $s_n \geq r - \frac{1}{n}$ and

$$A \models \vee_{t \in \mathbb{Q}' \cap [0, s_n]} R_{d(x,y) \geq t}(a, b) \wedge R_{d(x,y) \geq s_n - t}(b, c)$$

Thus, there is some $t_n \in \mathbb{Q}'$ such that $0 \leq t_n \leq s_n$ and

$$A \models R_{d(x,y) \geq t_n}(a, b) \wedge R_{d(x,y) \geq s_n - t_n}(b, c)$$

By the definition of D, this means that $D(a, b) \geq t_n$ and $D(b, c) \geq s_n - t_n$; thus, $D(a, b) + D(b, c) \geq s_n$. Since this is true for all $n < \omega$, we get that $D(a, b) + D(b, c) \geq r$ as desired.

Thus, D is a metric on $|A|$. $\dagger_{Claim}$

Now we define partial functions and relations on $(|A|, D)$ such that they are uniformly continuous. In particular,

1. for $i < L_F$, set $f_i := F_i^A$ with modulus $\Delta_{f_i}(r) = \sup\{s \in \mathbb{Q}' : A \vDash \forall x_0, \ldots, x_{n(F_i)-1}; \forall y_0, \ldots, y_{n(F_i)-1}$
$(\wedge_{i < n(F_i)} R_{d(z,z') \leq s}(x_i, y_i) \to R_{d(z,z') \leq r}(F_i(\mathbf{x}), F_i(\mathbf{y})))\}$.

2. for $j < L_R$, set $r_j(\mathbf{a}) := \sup\{r \in \mathbb{Q}' : A \vDash R_{R_j(\mathbf{z}) \leq r}[\mathbf{a}]\}$ with modulus $\Delta_{r_j}(r) = \sup\{s \in \mathbb{Q}' :$
$A \vDash \forall \mathbf{x} \forall \mathbf{y}(\wedge_{i < n(R_j)} R_{d(z,z') \leq s}(x_i, y_i) \to (R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}') \leq r}(\mathbf{x}, \mathbf{y}) \wedge R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}') \leq r}(\mathbf{y}, \mathbf{x})))\}$.

These functions are not defined on the desired structure (ie the completion of A), but they already fulfill our goal in terms of agreeing with the discrete relations in the following sense.

Claim: For all $\mathbf{a} \in A$ and all formulas $\phi(\mathbf{x})$ built up from these functions and D, we have that

$$\phi(\mathbf{a}) \square r \iff A \vDash R_{\phi(\mathbf{z}) \square r}[\mathbf{a}]$$

Proof: We proceed by induction on the construction of $\phi(\mathbf{x})$. We assume that $\square$ is $\geq$ in our proofs, but the proofs for $\leq$ are the same.

- If ϕ is atomic, then it falls into one of the following cases.

  - Suppose $\phi(\mathbf{x}) \equiv R_j(\tau(\mathbf{x}))$ for some term τ. Then

$$
\begin{aligned}
R_j^M(\tau(\mathbf{a})) \geq r &\iff \quad \inf\{s \in \mathbb{Q}' : A \vDash R_{R_j(\mathbf{x}) \geq s}[\tau(\mathbf{a})]\} \geq r \\
&\iff \quad \forall n < \omega, \exists s_n \in \mathbb{Q}' \text{ so } s_n \geq r - \frac{1}{n} \text{ and } A \vDash R_{R_j(\mathbf{x}) \geq s_n}[\tau(\mathbf{a})] \\
&\iff_{1d} \quad \forall n < \omega, A \vDash R_{R_j(\mathbf{x}) \geq r - \frac{1}{n}}[\tau(\mathbf{a})] \\
&\iff_{1g} \quad A \vDash R_{R_j(\mathbf{x}) \geq r}[\tau(\mathbf{a})] \\
&\iff_{2k} \quad A \vDash R_{R_j(\tau(\mathbf{y})) \geq r}[\mathbf{a}]
\end{aligned}
$$

  - Suppose that $\phi(\mathbf{x}, \mathbf{y}) \equiv d(\tau_1(\mathbf{x}), \tau_2(\mathbf{y}))$ for terms τ_1 and τ_2. The detail are essentially as above: $D^M(\tau_1(\mathbf{a}), \tau_2(\mathbf{b}))$ iff (by 1d, the definition of sup, and 1h and 1g) $M^+ \vDash R_{d(x,y) \geq r}[\tau_1(\mathbf{a}), \tau_2(\mathbf{b})]$ iff (by 2k) $M^+ \vDash R_{d(\tau_1(\mathbf{x}), \tau_2(\mathbf{y})) \geq r}(\mathbf{a}, \mathbf{b})$.

- For the inductive step, we deal with each connective (from our full set) in turn. The induction steps for $x \mapsto 0$, $x \mapsto 1$, and $x \mapsto \frac{x}{2}$ are clear.

  - Suppose $\phi \equiv \psi \dot{-} \tau$, where τ is a formula and not a term. Note if $r = 0$, then this is obvious. So assume $r \neq 0$. Recall that

$$
\phi^M(\mathbf{a}) = \psi^M(\mathbf{a}) \dot{-} \tau^M(\mathbf{a}) = \begin{cases} \psi^M(\mathbf{a}) - \tau^M(\mathbf{a}) & \text{if } \psi^M(\mathbf{a}) \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

Thus, we can assume we are in the case that $\psi^M(\mathbf{a}) > \tau^M(\mathbf{a})$.

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* First, suppose $\psi^M(\mathbf{a}) - \tau^M(\mathbf{a}) \geq r$. Since $\psi^M(\mathbf{a}) > \tau^M(\mathbf{a})$, there is some $s \in \mathbb{Q}'$ such that $\psi^M(\mathbf{a}) > s > \tau^M(\mathbf{a})$. Then $\tau^M(\mathbf{a}) \leq s$ and $\psi^M(\mathbf{a}) \geq s + r$. By induction, we have that
$$M^+ \models R_{\tau(\mathbf{x}) \leq s}[\mathbf{a}] \wedge R_{\psi^M(\mathbf{x}) \geq s + r}[\mathbf{a}]$$
Then, by 2e, we have that $M^+ \models R_{\psi \mathbin{\dot{-}} \tau \geq r}[\mathbf{a}]$ as desired.

* Now, suppose $M^+ \models R_{\psi \mathbin{\dot{-}} \tau \geq r}[\mathbf{a}]$. Again, 2e implies there is is some $s \in \mathbb{Q}'$ such that
$$M^+ \models R_{\tau \leq s}[\mathbf{a}] \wedge R_{\psi \geq r + s}[\mathbf{a}]$$
By induction, we get $\tau^M(\mathbf{a}) \leq s$ and $\psi^M(\mathbf{a}) \geq r + s$. Then
$$\phi^M(\mathbf{a}) = \psi^M(\mathbf{a}) - \tau^M(\mathbf{a}) \geq (r + s) - s = r$$

as desired.

- Suppose $\phi(\mathbf{x}) \equiv \sup_x \psi(x, \mathbf{x})$. We will consider both sides of the inequality since they're not symmetrically axiomatized (see 2g and 2h), but we won't worry about $\inf$.

  * Suppose that $\sup_x \phi^M(x, \mathbf{a}) \geq r$. Then for any $n < \omega$, there is some $a_n \in |M|$ such that $\phi^M(a_n, \mathbf{a}) > r - \frac{1}{2n}$. Since ϕ^M is uniformly continuous, there is some $\delta > 0$ such that, if $d(a_n, b) < \delta$, then $|\phi^M(a_n, \mathbf{a}) - \phi^M(b, \mathbf{a})| < \frac{1}{2n}$. Since M^+ is dense in M, there is some $a_n' \in M^+$ such that $d(a_n, a_n') < \delta$. Thus, $\phi^M(a_n', \mathbf{a}) > r - \frac{1}{n}$. By induction, we have that
  $$M^+ \models \wedge_{n<\omega} \exists x R_{\phi(y,\mathbf{y}) \geq r - \frac{1}{n}}(x, \mathbf{a})$$
  Then 2h says that $M^+ \models R_{\sup_y \phi(y,\mathbf{y}) \geq r}(\mathbf{a})$.

  * Suppose that $M^+ \models R_{\sup_y \phi(y,\mathbf{y}) \geq r}[\mathbf{a}]$. Then, by 2h, $M^+ \models \wedge_{n<\omega} \exists x R_{\phi(y,\mathbf{y}) \geq r - \frac{1}{n}}(x, \mathbf{a})$. So, for each $n < \omega$, there is some $a_n \in M^+$ such that $M^+ \models R_{\phi(y,\mathbf{y}) \geq r - \frac{1}{n}}[a_n, \mathbf{a}]$. By induction, we have that $\phi^M(a_n, \mathbf{a}) \geq r - \frac{1}{n}$. Since this is true for each $n < \omega$, we get $\sup_y \phi^M(y, \mathbf{a}) \geq r$.

  * The other direction is easier and we can combine the two parts

  $$\begin{aligned} \sup_x \phi^M(x, \mathbf{a}) \leq r \quad &\Longleftrightarrow \quad \forall a \in M \phi^M(a, \mathbf{a}) \leq r \\ &\Longleftrightarrow \quad \forall a \in M^+ \phi(a, \mathbf{a}) \leq r \\ &\Longleftrightarrow_{Induction} \quad M^+ \models \forall x R_{\phi \leq r}(x, \mathbf{a}) \\ &\Longleftrightarrow_{2g} \quad M^+ \models R_{\sup_x \phi(x,\mathbf{x})}[\mathbf{a}] \end{aligned}$$

  $\dagger$

We have given these functions moduli, but do not know they are uniformly continuous. We show this now. It is also worth noting that these moduli might not be the same moduli in the original signature L. Instead, these are the optimal moduli, while the original language might have moduli that could be improved.

Claim: The functions f_i and r_j are continuous.

Proof: We do each of these cases separately.

- **Sub-Claim 1:** $F_i^{M^+}$ is uniformly continuous on $(|M^+|, D)$ with modulus Δ_{F_i}.

  Let $r \in \mathbb{Q}'$ and let $\mathbf{a}, \mathbf{b} \in |M^+|$ such that $\max_{i<n} D(a_i, b_i) < \Delta_{F_i}(r)$. Thus, for each $i < n$, $D(a_i, b_i) = \inf\{s \in \mathbb{Q}' : M^+ \models R_{d(x,y)\leq s}(a_i, b_i)\} < \Delta_{F_i}(r)$. Since this is strict, there is some $s_i \in \mathbb{Q}'$ such that $M^+ \models R_{d(x,y)\leq s_i}(a_i, b_i)$. Note that 1d implies that the set $\Delta_{F_i}(r)$ is supremuming over is downward closed. Thus, $s' = \max_{i<n} s_i$ is in it. Thus, we can conclude

  $$M^+ \models R_{d(x,y)\leq r}[F_i(\mathbf{a}), F_i(\mathbf{b})]$$

  This means that $D(F_i(\mathbf{a}), F_i(\mathbf{b}) \leq r$, as desired.

- **Sub-Claim 2:** $R_j^{M^+}$ is uniformly continuous on $([0,1], |\cdot|)$ with modulus Δ_{R_j}.

  Let $r \in \mathbb{Q}'$ and $\mathbf{a}, \mathbf{b} \in |M^+|$ such that $\wedge_{i<n} D(a_i, b_i) < \Delta_{R_j}(r)$. From the infimum definition of D, for each $i < n$, there is $s_i \in \mathbb{Q}'$ such that $s_i < \Delta_{R_j}(r)$ and $M^+ \models R_{d(z,z')\leq s_i}[a_i, b_i]$. Thus,

  $$M^+ \models R_{R_j(\mathbf{z})\dot{-}R_j(\mathbf{z'})\leq r}(\mathbf{a}, \mathbf{b}) \wedge R_{R_j(\mathbf{z})\dot{-}R_j(\mathbf{z'})\leq r}(\mathbf{b}, \mathbf{a})$$

  For this next part, we need some of the future proofs, but essentially we have enough to show that this implies

  $$R_j^{M^+}(\mathbf{a})\dot{-}R_j^{M^+}(\mathbf{b}) \leq r \text{ and } R_j^{M^+}(\mathbf{b})\dot{-}R_j^{M^+}(\mathbf{a}) \leq r$$

  This implies $|R_j^{M^+}(\mathbf{a}) - R_j^{M^+}(\mathbf{b})| \leq r$, so $R_j^{M^+}$ is uniformly continuous. $\dagger$

Now we have a prestructure, see [BBHU08].§3. Now we complete $|A|$ to $|\bar{A}|$ in the standard way; see Munkries [Mun00] for a reference for the topological facts. In particular, we define the continuous L structure $\bar{A}$ by

- $|\bar{A}|$ is the completion of $(|A|, D)$;

- the metric $d^{\bar{A}}$ is the extension of D to $|\bar{A}|$;

- for $i < L_F$, $F_i^{\bar{A}}$ is the unique extension of f_i to $|\bar{A}|$; and

- for $j < L_R$, $R_j^{\bar{A}}$ is the unique extension of r_j to $|\bar{A}|$.

Essential inverses

Proposition 7.2.4. *Given any continuous L-structure M and dense subset A, we have that $M \cong_A \bar{M}_A$ and, given any L^+ structure A that models T_{dense}, we have that $(\bar{A})_A = A$.*

Proof: First, let M be a continuous L-structure and $A \subset |M|$ be nicely dense. We define a map $f : M \to \overline{(M_A)}$ as follows: if $a \in A$, then $f(a) = a$. For $a \in M - A$, fix some (any) sequence $\langle a_n \in A : n < \omega \rangle$ such that $\lim_{n\to\infty} a_n = a$ (this limit computed in M). We know that $\langle a_n : n < \omega \rangle$ is Cauchy in M, so it's Cauchy in $\overline{(M_A)}$. Then set $f(a) = \lim_{n\to\infty} a_n$, where that limit is computed in $\overline{(M_A)}$. This is well-defined and a bijection because A is dense in both sets. That this is an L-isomorphism follows from applying the correspondence twice: for all $\mathbf{a} \in A$ and $\phi(\mathbf{x}) \in Fml^c(L)$

$$\phi^M(\mathbf{a})\Box r \iff M_A \vDash R_{\phi(\mathbf{x})\Box r}[\mathbf{a}] \iff \phi^{\overline{(M_A)}}(\mathbf{a})\Box r$$

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and the fact that the values of ϕ on A determines its values on M and $(\bar{M}_A)$.

Second, let A be a L^+ structure that models T_{dense}. Clearly, the universes are the same, ie, $|(\bar{A})_A| = |A|$. For any relation $R_{\phi\square r}$ and $\mathbf{a} \in |A|$, we have

$$A \vDash R_{\phi\square r}[\mathbf{a}] \iff \phi^{\bar{A}}(\mathbf{a})\square r \iff (\bar{A})_A \vDash R_{\phi\square r}[\mathbf{a}]$$

Given a function F_i^+ and $\mathbf{a}, a \in A$, we have that

$$
\begin{aligned}
(F_i^+)^A(\mathbf{a}) = a \iff & A \vDash R_{d(F_i^+(\mathbf{x}),x)\leq 0}[\mathbf{a}, a] \\
\iff & (\bar{A})_A \vDash R_{d(F_i^+(\mathbf{x}),x)\leq 0}[\mathbf{a}, a] \iff (F_i^+)^{(\bar{A})_A}(\mathbf{a}) = a
\end{aligned}
$$

$$\dagger$$

We can extend this correspondence to theories. Suppose that T is a continuous theory in L. Following [BBHU08].4.1, theories are sets of closed L-conditions; that is, a set of "$\phi = 0$," where ϕ is a formula with no free variables. The following is immediate from Theorem 7.2.1.

Corollary 7.2.5. *If "$\phi = 0$" is a closed L-condition, then*

$$\phi^M = 0 \iff M_A \vDash R_{\phi\leq 0}$$

With our fixed theory T, set T^* to be $T_{dense} \cup \{R_{\phi\leq 0} : \text{``}\phi = 0'' \in T\}$. Then our representation of continuous L-structures as discrete L^+-structures modeling T_{dense} can be extended to a representation of continuous models of T and discrete models of T^*.

7.3 Elementary Substructure

We now discuss translating the notion of elementary substructure between our two contexts. Depending on the generality needed, this is either easy or difficult.

For the easy case, we have the following.

Theorem 7.3.1. *Let M, N be continuous L structures. Then $M \prec_L N$ iff, for every nicely dense $A \subset M$ and $B \subset N$ such that $A \subset B$, we have that $M_A \subset_{L^+} N_B$.*

Note that the relation between M_A and N_B is just substructure. So even though they are models of infinitary theories, their relation just concerns atomic formulas. This is because we have built the quantifiers of L into the relations of L^+.

Proof: $\leftarrow$: Let $A = M$ and $B = N$. Then $M \subset N$, so $M_M \subset_{L^+} N_N$ by assumption. Thus they agree on all relations concerning elements of M. Now we want to show that $M \prec_L^c N$. Let $\phi(\mathbf{x}) \in Fml^c L$ and $\mathbf{a} \in M$. From the theorems proved last section, we have, for each $r \in \mathbb{Q}'$,

$$
\begin{aligned}
\phi^M(\mathbf{a})\square r \iff & M_M \vDash R_{\phi(\mathbf{x})\square r}[\mathbf{a}] && \text{Theorem 7.2.1} \\
\iff & N_N \vDash R_{\phi(\mathbf{x})\square r}[\mathbf{a}] && M_M \subset_{L^+} N_N \\
\iff & \phi^N(\mathbf{a})\square r && \text{Theorem 7.2.1}
\end{aligned}
$$

Thus $\phi^M(\mathbf{a}) = \phi^N(\mathbf{a})$ and $M \prec_L N$ as desired.

$\rightarrow$: Let $A \subset M$ and $B \subset N$ be nicely dense so $A \subset B$. We want to show that $M_A \subset_{L^+} N_B$.

- Let $F^+ \in L^+$ and $\mathbf{a} \in A$. Then, by definition of the structures,

$$(F^+)^{M_A}(\mathbf{a}) = F^M(\mathbf{a}) = F^N(\mathbf{a}) = (F^+)^{N_B}(\mathbf{a})$$

- Let $R_{\phi\square r}(\mathbf{x}) \in L^+$ and $\mathbf{a} \in A$.

$$
\begin{aligned}
M_M \vDash R_{\phi(\mathbf{x})\square r}[\mathbf{a}] &\iff \phi^M(\mathbf{a})\square r && \text{Theorem 7.2.1} \\
&\iff \phi^N(\mathbf{a})\square r && M \prec N \\
&\iff N_N \vDash R_{\phi(\mathbf{x})\square r}[\mathbf{a}] && \text{Theorem 7.2.1}
\end{aligned}
$$

$\dagger$

Similarly, we have the following.

Theorem 7.3.2. *Given $A, B \vDash T_{dense}$, if $A \subset_{L^+} B$, then $\bar{A} \prec_L \bar{B}$.*

However, these are not the best theorems possible. In particular, the requirement that $A \subset B$ limits the scope of this theorem. We would like to know when L^+ structures complete to L-elementary substructures even when the dense substructures are not subsets or each other; for instance, the completions of $\mathbb{Q} \cap [0, 1]$ and $\mathbb{Q} + \sqrt{2}$ are nicely related, but the previous theorem does not see that. We would like to develop a criterion for L^+ structures $A, B \vDash T_{dense}$ that is equivalent to $\bar{A} \prec_L \bar{B}$.

Our first attempt is the following.

Theorem 7.3.3. *Suppose $M \prec N$ are continuous L-structures and $A \subset M$ and $B \subset N$ are nicely dense. Then*

1. *$M_A \subset_{L^+} N_C$, where C is the closure of $A \cup B$ under the functions of N.*

2. *There is a nicely dense $B' \subset N$ such that $M_A \subset_{L^+} N_{B'}$ and $|B'| = |A| + dc(N) + |L|$.*

Proof:

1. Note that $A \subset C$ and C is nicely dense in N. By Theorem 7.3.1, $M_A \subset_{L^+} N_C$.

2. Let $B'' \subset N$ be dense of size $dc(N)$ and let B' be the closure of $B'' \cup A$ under the functions of N. By Theorem 7.3.1, $M_A \subset_{L^+} N_{B'}$.

While this is an improvement, it is still not the best desireable. In particular, it still makes reference to the continuous structures. We would prefer a correspondence that only involved L^+ structures. To that end, we give the following definition of inessential extensions.

Definition 7.3.4. *Given $A \subset_{L_A^+} B$, we say that B is an* inessential extension *of A iff for every $b \in |B|$ and $n < \omega$, there is some $a \in |A|$ such that $B \vDash R_{d(x,y)<\frac{1}{n}}[b, a]$.*

Proposition 7.3.5. *If B is an inessential extension of A, then $\bar{A} = \bar{B}$.*

This gives us the following theorem.

Theorem 7.3.6. *Let $A, B \vDash T_{dense}$. Then TFAE*

1. $\bar{A} \prec_L \bar{B}$.

2. *There is an L^+ structure C such that $A, B \prec_{L_A^+} C$ and C is an inessential extension of B.*

3. *There is an L^+ structure $C' \vDash T_{dense}$ such that $A, B \subset_{L^+} C$ and C is an inessential extension of B.*

4. *There is an extension of the functions and relations of L^+ to $A \cup B$ such that $A \cup B \vDash T_{dense}$ that are still uniformly continuous and such that for every $a \in |A \cup B|$ and $n < \omega$, there is some $b \in |B|$ such that $A \cup B \vDash R_{d(x,y)<\frac{1}{n}}[b, a]$.*

Proof:

$\underline{(1) \implies (2)}$ Take $C = (\bar{B})_{A \cup B}$.

$\underline{(2) \implies (3)}$ Immediate.

$\underline{(3) \implies (4)}$ Take the extension inherited from C.

$\underline{(4) \implies (1)}$ We have $\bar{A}, \bar{B} \prec \overline{A \cup B}$ from the first condition and $\bar{B} = \overline{A \cup B}$ form the second.

7.4 Compactness and Ultraproducts

We pause here only briefly to point out a strange occurence: first-order continuous logic is compact (see [BBHU08].5.8), but $L_{\omega_1,\omega}$ is incompact. Yet, we have seen that continuous logic can be embedded into $L_{\omega_1,\omega}$, a seeming contradiction. The solution to this is that the compactness of continuous logic comes from a different ultraproduct than the model-theoretic one, namely the Banach space ultraproduct. In model theoretic terms, the Banach space ultraproduct avoids having elements of nonstandard norm by explicitly excluding all sequences with unbounded norm from the product. This is put into a general framework for type omission in the next chapter.

7.5 Types and Saturation

In this section, we will connect types in the continuous logic sense to types in the discrete sense. However, just as elements in the complete structure are represented by sequences of the discrete structure, we represent types by sequence types. Recall from [BBHU08].8.1 that a type over B is a collection of conditions of the form "$\phi(\mathbf{x}, \mathbf{b}) = 0$" with $\mathbf{b} \in B$.

Definition 7.5.1. • *We say that $\langle r_n : n < \omega \rangle$ is a sequence ℓ-type over B' iff $r_0(\mathbf{x})$ is an ℓ-type over B' and $r_{n+1}(\mathbf{x}, \mathbf{y})$ is a 2ℓ-type over B' such that there is some index set I, (possibly repeating) formulas $\langle \phi_i : i \in I \rangle$; and (possibly repeating) Cauchy sequences $\langle \langle \mathbf{b}_n^i \in B' \rangle_{n<\omega} : i \in I \rangle$ so $d(\mathbf{b}_n^i, \mathbf{b}_{n+1}^i) \leq \frac{1}{2^n}$ such that*

    – *$r_0(\mathbf{x}) = \{R_{\phi_i(\mathbf{z},\mathbf{z}') \leq w^{\phi_i}(2)}(\mathbf{x}, \mathbf{b}_0^i) : i \in I\}$; and*

    – *$r_{n+1}(\mathbf{x}, \mathbf{y}) = \{R_{\phi_i(\mathbf{z},\mathbf{z}') \leq w^{\phi_i}(\frac{1}{2^n})}(\mathbf{x}, \mathbf{b}_{n+1}^i) : i \in I\} \cup \{R_{d(z,z') \leq \frac{1}{2^n}}(x_k, y_k) : k < \ell\}$*

• *A realization of a sequence type $\langle r_n : n < \omega \rangle$ is $\langle \mathbf{a}_n : n < \omega \rangle$ such that*

– $\mathbf{a}_0$ *realizes* r_0*; and*

– $\mathbf{a}_{n+1}\mathbf{a}_n$ *realizes* r_{n+1}.

Note that the use of $\frac{1}{2^n}$ is not necessary; this could be replaced by any summable sequence for an equivalent definition (also replacing $\frac{1}{2^{n-1}}$ by the trailing sums). However, we fix $\frac{1}{2^n}$ for computational ease. The fundamental connection between continuous types and sequence types is the following.

Theorem 7.5.2. *Let $A \subset |M|$ be nicely dense.*

1. *If $B \subset |M|$ and $r(\mathbf{x})$ is a partial ℓ-type over B, then for any $B' \subset A$ such that $\bar{B}' \supset B$, there is a sequence ℓ type $\langle r_n : n < \omega \rangle$ over B' such that*

$$M \text{ realizes } r \text{ iff } M_A \text{ realizes } \langle r_n : n < \omega \rangle$$

2. *If $B' \subset A$ and $\langle r_n : n < \omega \rangle$ is a partial sequence ℓ-type over B', then there is a unique ℓ-type r over $\bar{B}'$ such that*

$$M \text{ realizes } r \text{ iff } M_A \text{ realizes } \langle r_n : n < \omega \rangle$$

We can denote the type in (2) by $\lim_{n \to \infty} r_n$. In each case, we have that $\langle \mathbf{a}_n \in M_A : n < \omega \rangle$ realizes $\langle r_n : n < \omega \rangle$ implies $\lim_{n \to \infty} \mathbf{a}_n$ realizes $\lim_{n \to \infty}$.

Proof:

1. Recall that $r(\mathbf{x})$ contains conditions of the form "$\phi(\mathbf{x}, \mathbf{b}) = 0$" for $\phi \in Fml^c(L)$ and $\mathbf{b} \in B$. For $n < \omega$ and $b \in B$, set $B'_n(b) = \{b' \in B' : d^M(b', b) < \frac{1}{2^n}\}$. To make the cardinality work out nicer, fix a choice function G, ie $G(B'_n(b)) \in B'_n(b)$. Then $B'_n(\mathbf{b})$ and $G(B'_n(\mathbf{b}))$ have the obvious meanings. Define

$$
\begin{aligned}
r_n^+(\mathbf{x}) &:= \{R_{\phi(\mathbf{z};\mathbf{y}) < w^\phi(\frac{1}{2^{n-1}})}(\mathbf{x}; G(B'_n(\mathbf{b}))) : \text{``}\phi(\mathbf{x}; \mathbf{b}) = 0\text{''} \in r\} \\
r_0(\mathbf{x}) &:= r_0^+(\mathbf{x}) \\
r_{n+1}(\mathbf{x}, \mathbf{y}) &:= r_{n+1}^+(\mathbf{x}) \cup \{R_{d(z,z') < \frac{1}{2^n}}(x_i, y_i) : i < \ell(\mathbf{x})\}
\end{aligned}
$$

Then $\langle r_n : n < \omega \rangle$ is a sequence type over B'; we can see this by taking r as the index set, $\phi_i = \phi$, and $\mathbf{b}^i = G(B'_n(\mathbf{b}))$ for $i = $ "$\phi(\mathbf{x}; \mathbf{b}) = 0$" $\in r$. To show it has the desired property, first suppose that $\langle \mathbf{a}_n : n < \omega \rangle$ from M_A realizes $\langle r_n : n < \omega \rangle$. We know that $\langle \mathbf{a}_n : n < \omega \rangle$ is a Cauchy sequence; in particular, for $m > n$,

$$d^M(\mathbf{a}_n, \mathbf{a}_m) \leq \sum_{i=n}^{m} \frac{1}{2^i} = \frac{2^{m+1-n} - 1}{2^{m+1}}$$

Since $\bar{M}_A \cong M$ is complete, there is $\mathbf{a} \in \bar{M}_A$ such that $\lim_{n \to \infty} \mathbf{a}_n = \mathbf{a}$. We claim that $\mathbf{a} \vDash r$. Let "$\phi(\mathbf{x}; \mathbf{b}) = 0$" $\in r$. Then

$$d^{\bar{M}_A}(\mathbf{ab}; \mathbf{a}_n G(B'_n(\mathbf{b}))) = \max\{d^M(\mathbf{a}, \mathbf{a}_n), d^M(\mathbf{b}, G(B'_n(\mathbf{b})))\} \leq \max\{\sum_{i=n}^{\infty} \frac{1}{2^i}, \frac{1}{2^n}\} = \frac{1}{2^{n-1}}$$

Thus,

$$|\phi^M(\mathbf{a}; \mathbf{b}) - \phi^M(\mathbf{a}_n; G(B'_n(\mathbf{b})))| < w^\phi(\frac{1}{2^{n-1}})$$

Letting $n \to \infty$, we have that

$$\phi^{\bar{M}_A}(\mathbf{a}; \mathbf{b}) = \lim_{n \to \infty} \phi^M(\mathbf{a}_n; G(B'_n(\mathbf{b}))) \leq \lim_{n \to \infty} w^\phi(\frac{1}{2^{n-1}}) = 0$$

as desired.

Now suppose that $\mathbf{a} \in M$ realizes r. Since A is dense, we can find $\langle \mathbf{a}_n \in A : n < \omega \rangle$ such that $d^M(\mathbf{a}_n, \mathbf{a}_{n+1}) < \frac{1}{2^n}$ and $\mathbf{a}_n \to \mathbf{a}$. We want to show that $r_n^+(\mathbf{a}_n)$ holds. Let "$\phi(\mathbf{x}, \mathbf{b})$" $\in r$. We know that

$$d^M(\mathbf{a}_n G(B'_n(\mathbf{b})), \mathbf{ab}) = \frac{1}{2^{n-1}}$$

, so we get

$$\phi^{M_A}(\mathbf{a}_n, G(B'_n(\mathbf{b}))) = |\phi^M(\mathbf{a}, \mathbf{b}) - \phi^M(\mathbf{a}_n, G(B'_n(\mathbf{b})))| < w^\phi(\frac{1}{2^{n-1}})$$

as desired. $\dagger$

2. Let $\langle r_n : n < \omega \rangle$ be a partial sequence ℓ-type given by I, $\langle \phi_i : i \in I \rangle$, and $\langle \langle \mathbf{b}_n^i \rangle_{n < \omega} : i \in I \rangle$. Then set

$$r(\mathbf{x}) := \{\phi_i(\mathbf{x}, \lim_{n \to \infty} \mathbf{b}_n^i) = 0 : i \in I\}$$

First, suppose that $\langle \mathbf{a}_n \in M_A : n < \omega \rangle$ realizes $\langle r_n : n < \omega \rangle$. Then, since $\mathbf{a}_{n+1} \mathbf{a}_n \vDash r_{n+1}$, we have $d^{M_A}(\mathbf{a}_{n+1}, \mathbf{a}_n) < \frac{1}{2^n}$ and, thus, the sequence is Cauchy. Since M is complete, let $\mathbf{a} = \lim_{n \to \infty} \mathbf{a}_n \in M$. Then, by uniform continuity, we have

$$\begin{aligned}
\phi_i^M(\mathbf{a}, \mathbf{b}^i) &= \phi_i^M(\lim_{n \to \infty} \mathbf{a}_n, \lim_{n \to \infty} \mathbf{b}_n^i) \\
&= \lim_{n \to \infty} \phi_i^M(\mathbf{a}_n, \mathbf{b}_n^i) \\
&\leq \lim_{n \to \infty} w^{\phi_i}(\frac{1}{2^{n-1}}) \\
&= 0
\end{aligned}$$

So $\mathbf{a} \vDash r$.

Now suppose that $\mathbf{a} \in M$ realizes t. Then, by denseness, we can find a Cauchy sequence $\langle \mathbf{a}_n \in A : n < \omega \rangle$ such that $d(\mathbf{a}_{n+1}, \mathbf{a}_n) \leq \frac{1}{2^n}$. Then $d(\mathbf{ab}^i, \mathbf{a}_n \mathbf{b}_n^i) \leq \frac{1}{2^{n-1}}$. Then we can conclude

$$|\phi_i^M(\mathbf{a}, \mathbf{b}^i) - \phi_i^M(\mathbf{a}_n, \mathbf{b}_n^i)| \leq w^{\phi_i}(\frac{1}{2^{n-1}})$$

$$\phi^{M_A}(\mathbf{a}_n, \mathbf{a}_n^i) \leq w^{\phi_i}(\frac{1}{2^{n-1}})$$

So $\langle \mathbf{a}_n : n < \omega \rangle$ realizes $\langle r_n : n < \omega \rangle$. $\dagger$

We now connect type-theoretic concepts in continuous logic (e.g. saturation and stability) with concepts in our discrete analogue.

Recall (see [BBHU08].7.5) that a continuous structure M is κ-saturated iff, for any $A \subset M$ of size $< \kappa$ and any continuous type $r(\mathbf{x})$ over A, if every finite subset of $r(\mathbf{x})$ is satisfiable in M, then so is $r(\mathbf{x})$.

Definition 7.5.3.   • *If $\langle r_n : n < \omega \rangle$ is a sequence type defined by an index set I and $I_0 \subset I$, then $\langle r_n : n < \omega \rangle^{I_0}$ is the sequence type defined by I_0.*

• *We say that $M_A \vDash T_{dense}$ is κ-saturated for sequence types iff, for all $B' \subset A$ and sequence type $\langle r_n : n < \omega \rangle$ over B' that is defined by I, if $\langle r_n : n < \omega \rangle^{I_0}$ is realized in M_A for all finite $I_0 \subset I$, then $\langle r_n : n < \omega \rangle$ is realized in M_A.*

Theorem 7.5.4.   1. *If M is κ-saturated and $\lambda^{\aleph_0} < \kappa$, then M_A is λ^+ saturated for sequence types.*

2. *If M_A is κ saturated, then M is κ saturated.*

Proof:

1. Let M be κ-saturated and $A \subset M$ be nicely dense. Let $B' \subset M_A$ of size λ and let $\langle r_n : n < \omega \rangle$ be a sequence type over B' that is finitely satisfiable in M_A. Set $r = \lim_{n \to \infty} r_n$ from Theorem 7.5.2; this is a type over $\bar{B}'$ where $|\bar{B}'| \leq \lambda^{\aleph_0} < \kappa$. We claim that r is finitely satisfiable in M. Any finite subset of r^- of

$$r = \{\phi_i(\mathbf{x}, \lim_{n \to \infty} \mathbf{b}_n^i) : i \in I\}$$

corresponds to a finite $I_0 \subset I$. Then, by Theorem 7.5.2, r_0 is realized in M iff $\langle r_n : n < \omega \rangle^{I_0}$ is realized in M_A. Then, since each $\langle r_n : n < \omega \rangle^{I_0}$ is realized in M_A by assumption, we have that r is finitely satisfiable in M. By the κ-saturation of M, r is realized in M. By Theorem 7.5.2, $\langle r_n : n < \omega \rangle$ is realized in M_A. So M_A is λ^+-saturated.

2. Let M_A be κ-saturated for sequence types. Let $B \subset M$ of size $< \kappa$ and r be a type over B that is finitely satisfied in B. Find $B' \subset A$ such that $\bar{B}' \supset B$; this can be done with $|B'| \leq |B| + \aleph_0 < \kappa$. Then form the sequence type $\langle r_n : n < \omega \rangle$ over B' that converges to r_n as in Theorem 7.5.2. As before, since r is finitely satisfiable in M, so is $\langle r_n : n < \omega \rangle$ in M_A. So $\langle r_n : n < \omega \rangle$ is realized in M_A by saturation. Thus, r is realized in M.   †

We immediately get the following corollary.

Corollary 7.5.5. *If $\kappa = (\lambda^{\aleph_0})^+$ or, more generally, $\kappa = \sup_{\lambda < \kappa} (\lambda^{\aleph_0})^+$ and M is of size κ, then M is saturated iff M_A is saturated for some nicely dense $A \subset M$ of size κ.*

7.6 T_{dense} **as an Abstract Elementary Class**

In this section we view the discrete side of things as an Abstract Elementary Class; see Baldwin [Bal09] or Grossberg [Gro1X].

Theorem 7.6.1. *Let T be a complete, continuous first order L-theory. Then let L^+ and T_{dense} be from Theorem 7.2.1. Set $K = (Mod\,(T_{dense} \cup T^*, \subset_{L^+})$. Then*

1. *K is an AEC;*

2. *K has amalgamation, joint embedding, and no maximal models; and*

3. *Galois types in K correspond to sequence types (Definition 7.5.1).*

Note that if T were not complete, then amalgamation would not hold. However, the other properties will continue to hold, including the correspondence between Galois types and sequence types.

Proof: $T_{dense} \cup T^+$ is a $L_{\omega_1, \omega}$ theory, so all of the examples hold except perhaps the chain axioms. For those, consider a $\subset_{L^+}$-increasing chain $\langle M_{A_i} : i < \alpha \rangle$. Then, by Theorem 7.2.1 and Theorem 7.3.2, the sequence $\langle \bar{M}_{A_i} : i < \alpha \rangle$ is $\prec_L$-increasing chain that each model T. Then by the chain axiom for continuous logic, there is $M = \cup_{i<\alpha} \bar{M}_{A_i}$ that models T. Additionally, $A := \cup_{i<\alpha} A_i$ is nicely dense in M. Thus, $M_A = \cup_{i<\alpha} M_{A_i}$ is as desired. Additionally, if $M_{A_i} \subset_{L^+} M_B$ for some B, then $M \prec_L \bar{M}_B$, so $M_A \subset_{L^+} M_B$.

These properties all follow from the corresponding properties of continuous first-order logic. For instance, considering amalgamation, suppose $M_A \subset_{L^+} M_B, M_C$. Then we have $\bar{M}_A \prec_L \bar{M}_B, \bar{M}_C$. By amalgamation for continuous first-order logic, there is some $N \succ_L \bar{M}_B$ and elementary $f : \bar{M}_C \to \bar{M}_A N$. Let $D \subset N$ be nicely dense that contains $B \cup C$. Then we have $M_B \subset_{L^+} N_D$ and $f \upharpoonright M_C : M_C \to_{M_A} N_D$; this is an amalgamation of the original system.

Finally, we wish to show that Galois types are sequence types and vice versa. Note that there are monster models in each class. Further more, we may assume that, if $\mathfrak{C}$ is the monster model of T, that there is some nicely dense $U \subset \mathfrak{C}$ such that the monster model of K is M_U; in fact, we could take $U = |\mathfrak{C}|$. Let continuous $M \vDash T$ and $A \subset M$ be nicely dense. If we have tuples $\mathbf{a}$ and $\mathbf{b}$, then

$$
\begin{aligned}
gtp_K(\mathbf{a}/M_A) = gtp_K(\mathbf{b}/M_A) \quad &\Longleftrightarrow \quad \exists f \in Aut_{M_A} M_U. f(\mathbf{a}) = \mathbf{b} \\
&\Longleftrightarrow \quad \exists f \in Aut_{\bar{M}_A} \mathfrak{C}. f(\mathbf{a}) = \mathbf{b} \\
&\Longleftrightarrow \quad tp(\mathbf{a}/M) = tp(\mathbf{b}/M) \\
&\Longleftrightarrow \quad \lim_{n \to \infty} r_n^{\mathbf{a}} = \lim_{n \to \infty} r_n^{\mathbf{b}}
\end{aligned}
$$

where $\langle r_n^x : n < \omega \rangle$ is the sequence type derived from $tp(x/M)$ as in Theorem 7.5.2 using A as the dense subset. $\dagger$

Chapter 8

A New Kind of Ultraproduct

8.1 Introduction

In this chapter, we provide a variant of the model-theoretic ultraproduct construction that, in some circumstances, can preserve type omission. Beyond giving this definition, the goal is to provide some sufficient conditions for when this behaves as desired and to demonstrate that there are strong consequences of this, mainly that the analogs results of Chapter IV and V hold in this context. Note however that this ultraproduct is defined entirely in ZFC as does not need any large cardinal hypotheses.

The motivation for this ultraproduct comes from the analysis of the previous chapter. The representation of continuous first order logic (which is compact) by $L_{\omega_1,\omega}$ (which is incompact) creates an obvious tension. As discussed there, the resolution to this tension is the Banach space ultraproduct. As detailed in Chapter II, that ultraproduct has two steps: only consider sequences that are bounded and to take those sequences modulo the U-limit of their distance. The second step has an analogue in the model-theoretic ultraproduct, although it is strengthened by some considerations specific to metric spaces. The first step does not. Essentially, it takes the type of an element that has infinite norm (i.e. one that realizes $p(x) = \{\|x\| > n : n < \omega\}$) and forces it to be omitted by excluding all sequences that would realize it. We generalize this procedure to an arbitrary model-theoretic setting with the Γ-ultraproduct.

Fix a collection of structures $\{M_i : i \in I\}$ in some fixed language L that all omit each 1-type from Γ. For ease, we enumerate $\Gamma = \{p^j(x) : j < \alpha\}$ and $p^j(x) = \{\phi_k^j(x) : k < \beta^j\}$, although the specific enumeration does not matter. In many natural examples, Γ is a single, countable type, but we develop a more general framework. The following definition outlines the crucial change from the classic ultraproduct.

Definition 8.1.1. •

$$\Pi_{i \in I}^\Gamma M_i := \{f \in \Pi M_i \quad : \quad \text{there is some } X_f \in U \text{ such that, for each } j < \alpha, \text{ there}$$
$$\text{is a } k_f^j < \beta^j \text{ such that } M_i \models \neg\phi_{k_f^j}^j[f(i)] \text{ for each } i \in X_f\}$$

Set $k(f) := \langle k_f^j : j < \alpha \rangle \in \Pi\beta^j$. Such a sequence is called a witness. Note that there might be several witnesses for a single function. If $\Gamma = \{p\}$, then we write $\Pi^p M_i$.

• Form $\Pi^\Gamma M_i/U$ by giving it universe $\Pi^\Gamma M_i/U := \{[f]_U : f \in \Pi^\Gamma M_i\}$ and inheriting the functions and relations from the full ultraproduct $\Pi M_i/U$.

Unfortunately, $\Pi^\Gamma M_i/U$ might fail to be an L-structure. That is, there is no reason that, in general $\Pi^\Gamma M_i/U$ is closed under L-functions, as shown by the following example.

Example 8.1.2. Set $M = (\omega, +, |, 2)$, $I = \omega$, $p(x) = \{(2^k \mid x) \wedge (x \neq 0) : k < \omega\}$. Then $[n \mapsto 1]_U, [n \mapsto 2^n - 1]_U \in \Pi^p M/U$, but

$$[n \mapsto 1]_U + [n \mapsto 2^n - 1]_U = [n \mapsto 2^n]_U \notin \Pi^p M/U$$

The above example shows that forcing the 2-adic valuation to be finite except on 0 does not work in this chapter. For this case, the next chapter provides a fix by returning to the Banach space ultraproduct.

In Section 8.2, we provide some criteria for when $\Pi^\Gamma M_i/U$ is an L-structure and when it satisfies Łoś' Theorem. In Section 8.3, we apply the results of previous chapters to this ultraproduct. Finally, Section 9.6 outlines some examples, although an example in ZFC where the full version of Łoś' theorem holds is not known at this time.

8.2   Properties of $\Pi^\Gamma M_i/U$

The following is a simple criterion for $\Pi^\Gamma M_i/U$ to be a structure.

Proposition 8.2.1. $\Pi^\Gamma M_i/U$ *is a structure iff*

for all $F \in L$, all $f_0, \ldots, f_{n-1} \in \Pi^\Gamma M_i$, there is some $X \in U$ such that, for all $j < \alpha$, there is some
$$k' := k^j_{F;f_0,\ldots,f_{n-1}} \text{ such that, for all } i \in X,$$

$$M_i \vDash \neg\phi^j_{k'}(F(f_0(i), \ldots, f_{n-1}(i)))$$

This is proved just by looking at the definition of what it means for $\Pi^\Gamma M_i$ to be closed under functions. We can also offer the following sufficient condition for when $\Pi^\Gamma M_i/U$ is an L-structure.

Definition 8.2.2. *The data is* okay *iff for all $j < \alpha$ and $F \in L$, there is a function $g^j_F : (\Pi\beta_{j'})^{n(F)} \to \beta_j$ such that, for all $i \in I$ and $a_0, \ldots, a_{n(F)-1} \in M_i$, we have $M_i \vDash \neg\phi^j_{k'}(F(a_0, a_{n(F)-1}))$ for $k' = g^j_F(k(a_0), \ldots, k(a_{n(F)-1}))$*

The g^j_F give a way to calculate where an element fails to realize a type based on how it is generated.

Proposition 8.2.3. *If the data is okay, then $\Pi^\Gamma M_i/U$ is an L-structure.*

If $\Pi^\Gamma M_i/U$ is a structure, then this is already enough to make a weak form of Łoś' Theorem hold.

Theorem 8.2.4 (Universal Łoś' Theorem). *Suppose $\Pi^\Gamma M_i/U$ is a structure. If $\phi(x_0, \ldots, x_n)$ is a universal formula and $[f_0]_U, \ldots, [f_{n-1}]_U \in \Pi^\Gamma M_i/U$, then*

$$\{i \in I : M_i \vDash \phi(f_0(i), \ldots, f_{n-1}(i))\} \in U \implies \Pi^\Gamma M_i/U \vDash \phi([f_0]_U, \ldots, [f_{n-1}]_U)$$

Proof: We go through formulas by induction and collect some results about implications between what holds in $\Pi^\Gamma M_i/U$ and what holds in a large set of M_i.

- **Atomic Formulas**

  Let R be an n-ary relation (possibly equality) and $\tau_0, \ldots, \tau_{n-1}$ be L-terms. Then

  $$(\tau_0^{\Pi^\Gamma M_i/U}([f_0]_U), \ldots, \tau_{n-1}^{\Pi^\Gamma M_i/U}([f_{n-1}]_U)) \in R^{\Pi^\Gamma M_i/U}$$
  $$\iff \{i \in I : (\tau_0^{\Pi^\Gamma M_i/U}([f_0]_U)(i), \ldots, \tau_{n-1}^{\Pi^\Gamma M_i/U}([f_{n-1}]_U)(i)) \in R^{M_i}\} \in U$$
  $$\iff \{i \in I : (\tau_0^{M_i}(f_0(i)), \ldots, \tau_{n-1}^{M_i}(f_{n-1}(i))) \in R^{M_i}\}$$

- **Conjunction/Disjunction**

  Suppose that $\phi = \psi\square\chi$, where $\square$ is $\wedge$ or $\vee$, and

  $$\Pi^\Gamma M_i/U \vDash \psi([f_0]_U, \ldots, [f_{n-1}]_U) \quad \diamond \quad \{i \in I : M_i \vDash \psi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$
  $$\Pi^\Gamma M_i/U \vDash \chi([f_0]_U, \ldots, [f_{n-1}]_U) \quad \diamond \quad \{i \in I : M_i \vDash \chi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$

where $\diamond$ is $\implies$, $\impliedby$, or $\iff$. Then

$$\Pi^\Gamma M_i/U \models \phi([f_0]_U, \ldots, [f_{n-1}]_U)$$
$$\iff \quad \Pi^\Gamma M_i/U \models \psi([f_0]_U, \ldots, [f_{n-1}]_U)\square\Pi^\Gamma M_i/U \models \chi([f_0]_U, \ldots, [f_{n-1}]_U)$$
$$\diamond \quad \{i \in I : M_i \models \psi(f_0(i), \ldots, f_{n-1}(i))\} \in U \square \{i \in I : M_i \models \chi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$
$$\iff \quad \{i \in I : M_i \models \psi(f_0(i), \ldots, f_{n-1}(i))\square\chi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$
$$\iff \quad \{i \in I : M_i \models \phi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$

- **Negation**

  Suppose that $\phi = \neg\psi$ and that

  $$\Pi^\Gamma M_i/U \models \psi([f_0]_U, \ldots, [f_{n-1}]_U) \quad \diamond \quad \{i \in I : M_i \models \psi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$

  where $\diamond$ is $\implies$, $\impliedby$, or $\iff$. Then

  $$\Pi^\Gamma M_i/U \models \phi([f_0]_U, \ldots, [f_{n-1}]_U)$$
  $$\iff \quad \Pi^\Gamma M_i/U \models \neg\psi([f_0]_U, \ldots, [f_{n-1}]_U)$$
  $$\iff \quad \Pi^\Gamma M_i/U \not\models \psi([f_0]_U, \ldots, [f_{n-1}]_U)$$
  $$reverse\diamond \quad \{i \in I : M_i \models \psi(f_0(i), \ldots, f_{n-1}(i))\} \notin U$$
  $$\iff \quad \{i \in I : M_i \models \neg\psi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$
  $$\iff \quad \{i \in I : M_i \models \phi(f_0(i), \ldots, f_{n-1}(i))\} \in U$$

- **Universal**

  Suppose that $\phi(\mathbf{x}) = \forall x \psi(x, \mathbf{x})$ such that

  $$\{i \in I : M_i \models \psi(f_0(i), \ldots, f_n(i))\} \in U \implies \Pi^\Gamma M_i/U \models \psi([f_0]_U, \ldots, [f_n]_U)$$

  and let $[f_0]_U, \ldots, [f_{n-1}]_U \in |\Pi^\Gamma M_i/U|$. Suppose $\{i \in I : M_i \models \phi(f_0(i), \ldots, f_{n-1}(i))\} \in U$ and let $[g]_U \in |\Pi^\Gamma M_i/U|$. Then

  $$\{i \in I : M_i \models \phi(f_0(i), \ldots, f_{n-1}(i))\} \in U \implies \{i \in I : M_i \models \forall x \psi(x, f_0(i), \ldots, f_{n-1}(i))\} \in U$$
  $$\implies \{i \in I : M_i \models \psi(g(i), f_0(i), \ldots, f_{n-1}(i))\} \in U$$
  $$\implies \Pi^\Gamma M_i/U \models \psi([g]_U, [f_0]_U, \ldots, [f_{n-1}]_U)$$

  Since $[g]_U$ was arbitrary, we get $\Pi^\Gamma M_i/U \models \phi([f_0]_U, \ldots, [f_{n-1}]_U)$

  $$\dagger$$

From this, we immediately get several corollaries and a type omission result for existential types.

Corollary 8.2.5. *Suppose $\Pi^\Gamma M_i/U$ is a structure. If $\phi(x_0, \ldots, x_n)$ is a quantifier-free formula and $[f_0]_U, \ldots, [f_{n-1}]_U \in \Pi^\Gamma M_i/U$, then*

$$\{i \in I : M_i \models \phi(f_0(i), \ldots, f_{n-1}(i))\} \in U \iff \Pi^\Gamma M_i/U \models \phi([f_0]_U, \ldots, [f_{n-1}]_U)$$

Proof: If ϕ is quantifier-free, then both ϕ and $\neg\phi$ are universal. Then we apply Theorem 8.2.4. $\qquad$ †

Corollary 8.2.6. *Suppose $\Pi^\Gamma M_i/U$ is a structure. If $\phi(x_0,\ldots,x_n)$ is an existential formula and $[f_0]_U,\ldots,[f_{n-1}]_U \in \Pi^\Gamma M_i/U$, then*

$$\Pi^\Gamma M_i/U \models \phi([f_0]_U,\ldots,[f_{n-1}]_U) \implies \{i \in I : M_i \models \phi(f_0(i),\ldots,f_{n-1}(i))\} \in U$$

Proof: If ϕ is existential, then $\neg\phi$ is universal. So

$$
\begin{aligned}
\Pi^\Gamma M_i/U \models \phi([f_0]_U,\ldots,[f_{n-1}]_U) &\implies \Pi^\Gamma M_i/U \not\models \neg\phi([f_0]_U,\ldots,[f_{n-1}]_U)\\
&\implies \{i \in I : M_i \models \neg\phi(f_0(i),\ldots,f_{n-1}(i))\} \notin U\\
&\implies \{i \in I : M_i \models \phi(f_0(i),\ldots,f_{n-1}(i))\} \in U
\end{aligned}
$$

$\qquad$ †

Proposition 8.2.7 (Weak Type Omission). *Suppose $\Pi^\Gamma M_i/U$ is an L-structure. If $j < \alpha$ and $p^j(x)$ consists of existential formulas, then $\Pi^\Gamma M_i/U$ omits p^j.*

Proof: Let $[f]_U \in \Pi^\Gamma M_i/U$ and $j < \alpha$. Then, by definition, for every $i \in X_f$, we have $M_i \models \neg\phi^j_{k_f^j}[f(i)]$. In particular, this is a U-large set. Since $\phi^j_{k_f^j}$ is existential, $\neg\phi^j_{k_f^j}$ is universal. So, by Theorem 8.2.4, we have

$$\Pi^\Gamma M_i/U \models \neg\phi^j_{k_f^j}([f]_U)$$

Thus $\Pi^\Gamma M_i/U$ omits p^j. $\qquad$ †

Ideally, the full version of Łoś' Theorem would hold if $\Pi^\Gamma M_i/U$ was a structure, but this might not be the case.

Example 8.2.8. *Let L be the two-sorted language $\langle N_1, N_2; +_1, \times_1, 1_1; +_2, \times_2, 1_2; \times_{1,2}\rangle$ where $\times_{1,2} : N_1 \times N_2 \to N_1$. Take $M = \langle \mathbb{N}, \mathbb{N}'; +, \times, 1; +', \times', 1'; \times^*\rangle$ where $\mathbb{N}$ and $\mathbb{N}'$ are disjoint copies of the naturals and $\times^*$ is also normal multiplication. Then this structure omits the type of a nonstandard element of the second sort $p(x) = \{N_2(x) \wedge (1 + \cdots + 1 \neq x) : n < \omega\}$ and models the sentence*

$$\phi \equiv \text{``}\forall x \in N_1 \exists y \in N_2 (1_1 \times_{1,2} y = x)\text{''}$$

Then $\Pi^p M/U$ is a structure. In particular, N_2 remains standard but N_1 is just $\Pi\mathbb{N}/U$. Thus, our sentence ϕ is no longer true and Łoś' Theorem must have failed.

However, we introduce the following condition that implies that Łoś' Theorem holds.

Definition 8.2.9. *The data is* strong *iff for all $j < \alpha$ and "$\exists x\phi(x; \mathbf{y})$" $\in L$, there is a function $g^j_{\exists x\phi(x;\mathbf{y})} : (\Pi\beta_{j'})^{\ell(\mathbf{y})} \to \beta_j$ such that, for all $i \in I$ and $a_0,\ldots,a_{\ell(\mathbf{y})-1} \in M_i$, we have*

if $M_i \models \exists x\phi(x; a_0,\ldots,a_{\ell(\mathbf{y})-1})$, then there is $b \in M_i$ such that $M_i \models \phi(b; a_0,\ldots,a_{\ell(\mathbf{y})-1})$ and
$$M_i \models \neg\phi^j_{k'}(b) \text{ for } k' = g^j_{\exists x\phi(x;\mathbf{y})}(k(a_0),\ldots,k(a_{\ell(\mathbf{y})-1}))$$

So being strong means that there is a uniform way to compute the places at which a witness to an existential fails to realize the types of Γ from the places at which the parameters fail to realize to types of Γ. The uniformity is not necessary for Łoś' Theorem, but is potentially helpful in applications.

The following facts are immediate:

Proposition 8.2.10. *If the data is strong, then the data is okay.*

    Proof Sketch: Take $g_F^j = g_{\exists x(x=F(\mathbf{y}))}^j$.   †

Proposition 8.2.11. *The data is strong iff there is a skolemization of the data that is okay.*

    Proof Sketch: Set $g_{\exists x\phi(x;\mathbf{y})}^j = g_{F_{\phi(x;\mathbf{y})}}^j$.   †

The main use of strongness is that it gives a sufficient condition for Łos' Theorem.

Theorem 8.2.12 (Łoś' Theorem)**.** *Suppose the data is strong. If $\phi(x_0,\ldots,x_n)$ is a formula and $[f_0]_U,\ldots,[f_{n-1}]_U \in \Pi^\Gamma M_i/U$, then*

$$\{i \in I : M_i \models \phi(f_0(i),\ldots,f_{n-1}(i))\} \in U \iff \Pi^\Gamma M_i/U \models \phi([f_0]_U,\ldots,[f_{n-1}]_U)$$

    Proof: By Proposition 8.2.10 and Theorem 8.2.4, all that needs to be shown is that adding an existential quantifier maintains transfer from "true in U-many M_i's" to "true in $\Pi^\Gamma M_i/U$." That is, suppose $\phi(\mathbf{y}) = \exists x\psi(x,\mathbf{y})$ such that

$$\{i \in I : M_i \models \psi(f_0(i),\ldots,f_n(i))\} \in U \implies \Pi^\Gamma M_i/U \models \psi([f_0]_U,\ldots,[f_n]_U)$$

We want to show that

$$X := \{i \in I : M_i \models \phi(f_0(i),\ldots,f_{n-1}(i))\} \in U \implies \Pi^\Gamma M_i/U \models \phi([f_0]_U,\ldots,[f_{n-1}]_U)$$

Let $g_{\exists x\psi(x;\mathbf{y})}^j$ from strength. By definition, for $i \in X$, there is some $b_i \in M_i$ such that

- $M_i \models \psi[b_i; f_0(i),\ldots,f_{n-1}(i)]$; and

- $M_i \models \neg\phi_{k'}^j(b_i)$ for $k' = g_{\exists x\phi(x;\mathbf{y})}^j(k(f_0),\ldots,k(f_{n-1}))$.

Define $h \in \Pi M_i$ by

$$h(i) = \begin{cases} b_i & \text{if } i \in X \\ arb. & \text{if } i \notin X \end{cases}$$

    Now we claim that $h \in \Pi^\Gamma M_i$ since $k(h) = \langle g_{\exists x\phi(x;\mathbf{y})}^j(k(f_0),\ldots,k(f_{n-1})) : j < \alpha \rangle$ is a witness on X.

    Then

$$\begin{aligned}
\{i \in \quad I \quad : \psi(h(i); f_0(i),\ldots,f_{n-1}(i))\} &= X \in U \\
\Pi^\Gamma M_i/U \quad &\models \quad \psi([h]_U; [f_0]_U,\ldots,[f_{n-1}]_U) \\
\Pi^\Gamma M_i/U \quad &\models \quad \phi([f_0]_U,\ldots,[f_{n-1}]_U)
\end{aligned}$$

as desired.   †

Once we have the full strength of Łoś' Theorem, we are guaranteed the the end structure omits the desired types.

Proposition 8.2.13 (Type Omission)**.** *Suppose the data is strong (or just Łoś' Theorem holds). Then* $\Pi^\Gamma M_i/U$ *omits each type in* Γ.

Proof: Let $[f]_U \in \Pi^\Gamma M_i/U$. This is witnessed by $k(f) = \langle k_f^j : j < \alpha \rangle$. Fix $j < \alpha$. Then

$$\{i \in I : M_i \vDash \neg\phi_{k_f^j}^j(f(i))\} = X_f \in U$$

By Theorem 8.2.12, this means

$$\Pi^\Gamma M_i/U \vDash \neg\phi_{k_f^j}^j([f]_U)$$

So every element of $\Pi^\Gamma M_i/U$ does not realize any type from Γ.   †

Summarizing our results so far, we have the following.

Corollary 8.2.14. *If the data is strong, then* $\Pi^\Gamma M_i/U$ *is an L-structure that satisfies Łoś' Theorem and omits every type in* Γ*. In particular, if* $M_i \in EC(T, \Gamma)$ *for all* $i \in I$*, then* $\Pi^\Gamma M_i/U \in EC(T, \Gamma)$*.*

Proof: By Theorems 8.2.12 and 8.2.13.   †

We now turn our attention to ultrapowers, where $M_i = M$ for all $i \in I$. In this case, set $up : M \to \Pi^\Gamma M_i/U$ to be the ultrapower map by $up(m) = [i \mapsto m]_U$. This is a function even if $\Pi^\Gamma M_i/U$ and Łoś' Theorem is equivalent to up being an elementary embedding. We would also like to know when this construction gives rise to a new model. Unfortunately, this is not always the case.

Example 8.2.15. *Let* U *be an ultrafilter on* I*. Take the ultraproduct of* $\mathbb{N} = \langle \omega, +, \cdot, < \rangle$ *omitting* $p(x) = \{x > n : n < \omega\}$*; then* $up : \mathbb{N} \cong \Pi^p\mathbb{N}/U$*.*

Proof: First, we need to show that this Data is strong. It is obviously weak: $g_+(x, y) = x + y$ and $g_\cdot(x, y) = x \cdot y$. This is enough to make the conclusion well-formed (ie $\Pi^p\mathbb{N}/U$ is a structure), so we omit the details of strength. If $f \in \Pi^p\mathbb{N}$, then there is some $k_f < \omega$ such that $f(i) < k_f$ for all $i \in I$. Since U is ω-complete (as are all ultrafilters) and k_f is finite, there is some $n_f < k_f$ such that $\{i \in I : f(i) = n_f\} \in U$. Thus, $[f]_U = [i \mapsto n_f]_U$ and the mapping $h : \Pi^p\mathbb{N}/U \to \mathbb{N}$ by $h([f]_U) = n_f$ is an isomorphism.   †

This did not give rise to a new model because the value of $k(f)$, here a single natural number, determined which element of $\mathbb{N}$ the function represented. In order to insure that up is not surjective, we need to ensure that there are many choices that give rise to the same $k(f)$.

Theorem 8.2.16. *Suppose* M *is a model omitting* Γ *and there is some* $\langle k_j < \beta^j : j < \alpha \rangle$ *and infinite* $X \subset |M|$ *such that for every* $x \in X$ *and* $j < \alpha$*, we have* $M \vDash \neg\phi_{k_j}^j(x)$*. Then up is not surjective onto* $\Pi^\Gamma M/U$*.*

Proof: Let U be a nonprincipal ultrafilter on ω. Then let $f : \omega \to X$ enumerate distinct members of X. By definition of X, $f \in \Pi^\Gamma M$. Since U is nonprincipal, $f \not\!\!U g$ for any constant $g \in \Pi^\Gamma M$. Thus, $[f]_U$ is an extra element in $\Pi^\Gamma M/U$. †

Corollary 8.2.17. *If* $\|M\| > |\Pi_{j<\alpha}\beta^j|$, *then up is not surjective.*

Proof: For each $x \in M$, pick $k^x = \langle k_j^x : j < \beta \rangle$ such that $M \vDash \neg\phi_{k_j^x}^j(x)$. There are $|\Pi\beta^j|$ many possible values for k^x. Since $\|M\|$ is greater than this, there must be some infinite $X \subset |M|$ such that the choice is constant. Then apply Theorem 8.2.16. †

Corollary 8.2.18. *Suppose p is countable and M is uncountable. If U is nonprincipal, then $\Pi^p M/U \not\cong M$.*

8.3   Abstract Elementary Classes

Definition 8.3.1. *An AEC K is* nice *when*

1. *$K = EC(T, \Gamma)$ for some theory T, set of types $\Gamma = \{p^j(x) : j < \alpha\}$ with $p^j(x) = \{\phi_k^j(x) : k < \beta^j\}$;*

2. *$\prec_K = \prec_{L(T)}$; and*

3. *if U is an ultrafilter on an index set I and $\{M_i \in K : i \in I\}$, then this data satisfies Łoś' Theorem.*

Note that this does not depend on the enumeration given.

The first consequence of strength is proper extensions.

Proposition 8.3.2. *Suppose K is nice. Then $K_{|\Pi\beta^j|}$ has no maximal models.*

This immediately follows from Theorem 8.2.16 and the corollary that follows it.

This result gives the first strength of a class being closed under some notion of ultraproduct: the ability to create new models. However, ultraproducts go beyond this; they allow the construction of a new model whose properties are, in a sense, the average of the properties of some other model. We use the ultraproducts to emulate the locality results of Chapter IV and to develop an independence relation as in Chapter V. Because the ultrafilter is only required to be ω-complete, we get the best possibly locality results.

Theorem 8.3.3. *Suppose K is strong. Then K is fully $LS(K)$-tame and fully $< \omega$-type short.*

Proof: We prove the second part and note that it implies the first by Theorem 2.2.6. Suppose that $X = \langle x_i \in M_1 : i \in I \rangle$ and $Y = \langle y_i \in M_2 : i \in I \rangle$ are given such that, for all $I_0 \in P_\omega I$,

$$gtp(\langle x_i : i \in I_0 \rangle/\emptyset; M_1) = gtp(\langle y_i : i \in I_0 \rangle/\emptyset; M_2)$$

That is, there is $N_{I_0} \in K$ and $f_{I_0}^\ell : M_\ell \to N_{I_0}$ such that $f_{I_0}^1(x_i) = f_{I_0}^2(y_i)$ for all $i \in I_0$. Let U be a fine ultrafilter on $P_\omega I$. Then, following Chapter IV, set

- $N = \Pi^{\Gamma}_{I_0 \in P_\omega I} N_{I_0}/U$;

- $f^\ell : M_\ell \to N$ is given by $f^\ell(m) = [I_0 \mapsto f^\ell_{I_0}(m)]_U$

N is well-defined by hypothesis and f^ℓ is a K-embedding by Lemma 8.3.4 below. For each $i \in I$, $\{I_0 \in P_\omega I : f^1_{I_0}(x_i) = f^2_{I_0}(y_i)\}$ contains $[i] := \{I_0 \in P_\omega I : i \in I_0\} \in U$ by the fineness. So $f^1(x_i) = f^2(y_i)$ for all $i \in I$. Then

$$gtp(X/\emptyset; M_1) = gtp(Y/\emptyset; M_2)$$

$\dagger$

Lemma 8.3.4. *Suppose that $\langle M_i : i \in I \rangle$ and $\langle N_i : i \in I \rangle$ and $f_i : M_i \to N_i$ is elementary. Then $f : \Pi^{\Gamma} M_i/U \to \Pi^{\Gamma} N_i/U$ by $f([i \mapsto m_i]_U) = [i \mapsto f_i(m_i)]_U$ is elementary.*

Proof: First, we need to know that $[i \mapsto f_i(m_i)]_U$ is in $\Pi^{\Gamma} N_i/U$. This is true because, by the elementarity of each f_i,

$$M_i \vDash \neg\phi^j_k(m_i) \implies N_i \vDash \neg\phi^j_k(f_i(m_i))$$

So $k([i \mapsto m_i]_U)$ is a witness for $[i \mapsto f_i(m_i)]_U$. Thus, Łos' Theorem 8.2.12 shows that f is elementary.$\dagger$

We could repeat the proofs if we knew there were more complete ultrafilters.

Theorem 8.3.5. *Suppose K is strong for κ-complete ultrafilters and κ is measurable. Then K is $(< \lambda, \lambda)$-tame and -type short for cf $\lambda = \kappa$.*

Theorem 8.3.6. *Suppose K is strong for κ-complete ultrafilters and κ is strongly compact. Then K is fully $< \kappa$-tame and -type short.*

Now, following Chapter V, we define the following notion of coheir.

Definition 8.3.7. *Given $M \prec N \prec \widehat{M}$ and $A \subset \widehat{M}$, we say that $A \underset{M}{\overset{\widehat{M}}{\downarrow}} N$ iff*

for all $N^- \prec N$ of size $LS(K)$ and $A^- \subset A$ of size $LS(K)$, $gtp(A^-/N^-; \widehat{M})$ is realized in M.

This ultraproduct allows us to weaken the requirements on getting this to be an independence relation the same way as in Chapter V.§6.

Theorem 8.3.8. *Suppose that K is a strong AEC with amalgamation and joint embedding. If K has no order property and satisfies Existence, then $\downarrow$ is an independence relation.*

8.4 Partial Examples

In this section, we discuss some examples.

Banach spaces are our prototypical example and the motivating example for this work, flowing from the representation of model theory for metric structures as a certain infinitary fragment in the last chapter. We outline how this can be put into this framework.

Let $L_b = \langle B, R; +_B, 0_B; +_R, \cdot_R, 0_R, 1_R, <_R, c_r; \| \cdot \|, \cdot_{scalar} \rangle_{r \in \mathbb{R}}$ be the two sorted language of normed linear spaces. Then T_b says that

- R is a copy of $\mathbb{R}$; and

- B is a vector space over R, with norm $\|\cdot\| : B \to R$.

We want to ensure that, in the ultraproduct, R and F each have no nonstandard elements. Note that ensuring just one of these is not enough as we might run into issues of the ultraproduct not being a structure. Similarly, we cannot forbid infinite elements and rely on the field structure to imply there are no infinitesimals. Instead, we have to forbid each possibility explicitly. Thus, we wish to omit the following types. We drop the subscripts on the language for notational ease.

- $p_\infty(x) = \{R(x) \wedge (x < -n \vee n < x) : n < \omega\}$;

- $p_r(x) = \{R(x) \wedge (x \neq c_r) \wedge (c_{r-\frac{1}{n}} < x < c_{r+\frac{1}{n}}) : n < \omega\}$ for $r \in \mathbb{R}$;

- $q_\infty(x) = \{B(x) \wedge (\|x\| < -n \vee n < \|x\|) : n < \omega\}$; and

- $q_r(x) = \{B(x) \wedge (\|x\| \neq c_r) \wedge (c_{r-\frac{1}{n}} < x < c_{r+\frac{1}{n}}) : n < \omega\}$.

Set $\Gamma = \{p_r(x) : r \in \mathbb{R} \cup \{\infty\}\} \cup \{q_r(x) : r \in \mathbb{R}^{\geq 0} \cup \{\infty\}\}$. We claim that $\Pi^\Gamma \mathcal{B}_i / U$ is an L_b-structure for any $\mathcal{B}_i$ and ultrafilter U. We omit the details, but standard real number that two sequences correspond to, which are their witnesses to inclusion in $\Pi^\Gamma \mathcal{B}_i$, can be used to calculate which number their sum or product corresponds to. Thus, it is closed under functions and satisfies the Universal Łoś' Theorem.

The next example is abelian torsion groups. Let $L_g = \{+, 0, -\cdot\}$ and T_{ag} be the theory of abelian groups. Abelian torsion groups are models of T_{ag} that omit $p(x) = \{n \cdot x \neq 0 : n < \omega\}$. We claim that abelian torsion groups are okay.

Suppose we have the following data.

- I, an index set;

- U, an ultrafilter on I;

- $L_g = \{+, 0, -\}$, the language of groups;

- $p(x) = \{n \cdot x \neq 0 : n < \omega\}$, the type of an element with infinite order;

- $\{G_i : i \in I\}$, a set of abelian torsion groups.

Proposition 8.4.1. *This data is okay.*

Proof: Given $g \in G_i$, we have that $G_i \vDash \neg(n \cdot g \neq 0)$ exactly when $o(g) \mid n$. Since $o(g) = o(-g)$ and $o(g_1 + g_2) = \mathrm{lcm}(o(g_1), o(g_2)) \mid o(g_1)o(g_2)$, setting $g_-(n) = n$ and $g_+(n, m) = nm$ shows the data is okay. $\dagger$

Our final example shows that, if there are very complete ultraproducts, then this new ultraproduct coincides with the classic one.

Theorem 8.4.2. *If U is χ-complete and $\chi > \beta^j$ for all $j < \alpha$ and $\chi > \alpha$, then $\Pi^\Gamma M_i / U = \Pi M_i / U$.*

Proof: We always have $\Pi^\Gamma M_i \subset \Pi M_i$. Let $f \in \Pi M_i$. We want to show $f \in \Pi^\Gamma M_i$ by finding a witness. For each $j < \alpha$, set $X_k^{f,j} := \{i \in I : M_i \vDash \neg \phi_k^j(f(i))\}$. Then $\{X_k^{f,j} : k < \beta^j\}$ is a partition of I into β^j many pieces. Since $\beta^j < \chi$, there is some $k_j < \beta^j$ such that $X_{k_j}^{f,j} \in U$. Then

$$X^f = \cap_{j<\alpha} X_{k_j}^{f,j} \in U$$

shows that $k(f) = \langle k_j : j < \alpha \rangle$ is a witness. Thus $\Pi^\Gamma M_i = \Pi M_i$. †

Chapter *9*

Some Model Theory of Classically Valued Fields

9.1 Introduction

In this chapter, we begin to explore the model theory of classically valued fields. These are fields that are equipped with a map (called a valuation) into $\mathbb{R}$ that satisfies certain axioms; the prototypical example is the p-adic valuation on $\mathbb{Q}$ that counts divisibility by p. The adjective "classic" (sometimes "rank 1") denotes that the range of the valuation (called the value group) is a subset of $\mathbb{R}$ contrasts with Krull valuations, where the the value group is allowed to be an arbitrary ordered abelian group. Note that the condition that the valuation maps to $\mathbb{R}$ is equivalent to requiring that the value group be Archimedean.

The model theory of Krull valuations has been well-studied; van den Dries [vdD] and Haskell, Hrushovski, and Macpherson [HHM08] provide good refeerences. The focus on Krull valuations by model-theorists is due to the fact that they are first-order axiomatizable and the theory becomes complete with the additional specification that the field is algebraically closed and the choice of the characteristic for the field and residue field. However, the methods used are not always applicable to classical valuations because the Archimedean property requires infinitary logic to express.

Instead, we use some AEC machinery developed in the previous chapters, especially Chapters IV and V, to analyze it. In particular, the workhorse of this analysis is an ultraproduct construction. The model-theoretic ultraproduct will not work; it is well known that the value group of an ultraproduct is the ultraproduct of the value groups and, thus, will fail to be Archimedean in an interesting case. Instead, we use the ultraproduct from analysis described in Chapter II. This ultraproduct is well studied in analysis in the context of Banach spaces and metric spaces and, as mentioned in Chapter VII, is the ultraproduct used for continuous logic. In fact, Ben Yaacov [BY] has previously studied classically valued fields in the continuous logic. However, he does so by passing to the projective line in order to avoid unbounded spaces, while we are able to work with the valued field directly. More generally, we avoid continuous logic because we don't fit within its framework (we deal with unbounded spaces, we don't require the spaces to be complete, etc.) and to work with true/false valued formulas.

We begin by reviewing some facts from valued field theory. Then we give the ultraproduct construction in Definition 9.3.4 and prove some basic transfer facts; although this construction is not new, we reprove some results in this context for completeness. Section 9.6 computes some examples of ultraproducts of p-adics to familiarize the reader with this construction. Section 9.7 begins the model-theoretic analysis by using quantifier elimination results of Robinson [Rob56] to prove a variant of Łoś' Theorem.

9.2 Preliminaries

A good reference for classical valuations is Ribenboim's book [Rib99]. The model theoretic references given above ([vdD] and [HHM08]) also review some valuation theory, but typically for the context of Krull valuations.

There are three equivalent ways to develop valuation theory: absolute values, valuations, or valuation divisibilities. We work with absolute values, but the other definitions would work as well. More precisely, given a non-Archimedean[1] absolute value $|\cdot|$, this gives rise to a valuation $v(x) := -\ln|x|$ and a valuation divisibility $|$ by $x \mid y$ iff $v(x) \leq v(y)$.

Definition 9.2.1. • *Given a commutative ring with unity K, $|\cdot| : K \to \mathbb{R}$ is an* absolute value *iff*

---

[1]It is crucial to distinguish the notion of an *ordering* being Archimedean from the notion of an *absolute value* being Archimedean, which means it is equivalent to the standard absolute value or that it doesn't satisfy the ultrametric property

I. $|x| = 0$ *iff* $x = 0$;

II. $|xy| = |x||y|$; *and*

III. $|x + y| \leq |x| + |y|$.

- *An absolute value is called* non-Archimedean *iff III. can be strengthened to*

III'. $|x + y| \leq \max\{|x|, |y|\}$.

The most basic example of an absolute value is the standard absolute value on $\mathbb{R}$. However, this is Archimedean and, thus, corresponds to no valuation. The most basic non-Archimedean example is the p-adics: fix a prime p and, for $x \in \mathbb{Z}$ set $n_p(x)$ to be the number of times p divides x. Then the p-adic absolute value on $\mathbb{Q}$ is $|\frac{a}{b}|_p = p^{-n_p(a) + n_p(b)}$ with $|0|_p = 0$. This gives rise to inequivalent absolute values for each p and, with the standard absolute value, characterizes all nontrivial absolute values on $\mathbb{Q}$.

9.3 Basic Construction

For each $i \in I$, let $(K_i, |\cdot|_i)$ be a commutative ring with unity with an absolute value. Eventually, K_i will be an algebraically closed field and $|\cdot|_i$ will be a nontrivial, non-Archimedean absolute value, but we delay this until necessary.

Definition 9.3.1. *Set* $\Pi^* K_i := \{f \in \Pi K_i : \exists n_f. \forall i \in I, |f(i)|_i < n_f\}$.
For $f, g \in \Pi^* K_i$, *set*

$$f U^* g \text{ iff for all } k < \omega, \ X_{f,g}^k := \{i \in I : |f(i) - g(i)|_i < \tfrac{1}{k}\} \in U.$$

In fact, it is equivalent to require just that there are unboundedly many $k < \omega$ such that $X_{f,g}^k \in U$. We wish to take the functions $\Pi^* K_i$ modulo the equivalence relation U^*. However, we must first prove that U^* is indeed an equivalence relation and, moreover, a congruence relation for the field operations and the absolute value.

Claim 9.3.2. U^* *is an equivalence relation on* $\Pi^* K_i$.

Proof: Clear. †

Claim 9.3.3. *Suppose* $f_\ell, g_\ell \in \Pi^* K_i$ *for* $\ell < 2$ *such that* $f_0 U^* f_1$ *and* $g_0 U^* g_1$. *Then*

1. $(f_0 \pm g_0), f_0 g_0 \in \Pi^* K_i$;

2. $(f_0 \pm g_0) U^* (f_1 \pm g_1)$;

3. $f_0 g_0 U^* f_1 g_1$; *and*

4. $\lim_U |f_0(i)|_i = \lim_U |f_1(i)|_i$.

Proof:

1. For all $i \in I$,
$$|f_0(i) \pm g_0(i)|_i \leq |f_0(i)|_i + |g_0(i)|_i$$

so taking $n_{f_0 \pm g_0} = n_{f_0} + n_{g_0}$ witnesses $f_0 \pm g_0 \in \Pi^* K_i$. Similarly,

$$|f_0(i)g_0(i)|_i = |f_0(i)|_i |g_0(i)|_i$$

so $n_{f_0 g_0} = n_{f_0} n_{g_0}$ witnesses $f_0 g_0 \in \Pi^* K_i$.

2. For all $i \in I$,
$$|(f_0(i) \pm g_0(i)) - (f_1(i) \pm g_1(i))|_i \leq |f_0(i) - f_1(i)|_i + |g_0(i) - g_1(i)|_i$$

Thus,
$$X^k_{f_0,g_0} \cap X^k_{f_1,g_1} \subset X^{2k}_{f_0 \pm g_0, f_1 \pm g_1} \in U$$

So $f_0 \pm g_0 U^* f_1 \pm g_1$.

3. Fix $k < \omega$. For all $i \in X^k_{f_0,g_0} \cap X^k_{f_1,g_1}$,

$$\begin{aligned}
|f_0(i)g_0(i) - f_1(i)g_1(i)|_i &= |f_0(i)g_0(i) - f_0(i)g_1(i) + f_0(i)g_1(i) - f_1(i)g_1(i)|_i \\
&\leq |f_0(i)|_i |g_0(i) - g_1(i)|_i + |g_1(i)|_i |f_0(i) - f_1(i)|_i \\
&\leq n_{f_0} \frac{1}{k} + n_{g_1} \frac{1}{k} = \frac{n_{f_0} + n_{g_1}}{k}
\end{aligned}$$

So, for any $l < \omega$,
$$X^l_{f_0,f_1} \cap X^l_{g_0,g_1} \subset X^{(n_{f_0} + n_{g_1})l}_{f_0 g_0, f_1 g_1} \in U$$

So $f_0 g_0 U^* f_1 g_1$.

4. Note that $0 \leq |f_\ell(i)|_i \leq n_f$ for all $i \in I$, so the ultrapower limit exists. Then $f_\ell U^* f_{1-\ell}$, so $\lim_U |f_\ell(i) - f_{1-\ell}(i)|_i = 0$. Since $|\cdot|_i$ is an absolute value,

$$\begin{aligned}
0 &= \lim_U |f_\ell(i) - f_{1-\ell}(i)|_i \\
&\geq \lim_U |f_\ell(i)|_i - \lim_U |f_{1-\ell}(i)|_i
\end{aligned}$$

So $\lim_U |f_\ell(i)|_i \geq \lim_U |f_{1-\ell}(i)|_i$. So $\lim_U |f_0(i)|_i = \lim_U |f_1(i)|_i$.   †

By the above, we can make the following definitions and they are well-defined.

Definition 9.3.4. *We define the structure* $(\Pi^* K_i / U^*, |\cdot|_*)$ *by*

- *the universe is* $\{[f]_{U^*} : f \in \Pi^* K_i\}$;

- $[f]_{U^*} \pm [g]_{U^*} = [f \pm g]_{U^*}$;

- $[f]_{U^*}[g]_{U^*} = [fg]_{U^*}$; *and*

- $|[f]_{U^*}|_* = \lim_U |f(i)|_i$.

9.4 Properties of $\Pi^* K_i / U^*$

We now explore the properties of $\Pi^* K_i / U^*$. As expected, it inherits its properties from those of the K_i. Some results below are phrased as "If each K_i is P, then..."; each of these could be stated as "If there is $X \in U$ such that K_i has property P for all $i \in X$, then ...."

Claim 9.4.1. $\Pi^* K_i / U^*$ *is a commutative ring with unity.*

Proof: The commutative ring part holds by definition of $+$ and $\cdot$. Note that $|1_i|_i$ by the definition of absolute value, so $[1_i]_{U^*} \in \Pi^* K_i / U^*$. †

Claim 9.4.2. *If each K_i is a field, then $\Pi^* K_i / U^*$ is a field.*

Proof: By above, $\Pi^* K_i / U^*$ is a commutative ring with unity. Let $[f]_U^* \neq [0]_{U^*}$. Since $f \not{U^*} 0$, there is some $k_f < \omega$ such that $\{i \in I : |f(i)|_i < \frac{1}{k_f}\} \notin U$. Thus,

$$X := \{i \in I : |f(i)|_i \geq \frac{1}{k_f}\} \in U$$

So, for all $i \in X$, we have

$$\frac{1}{k_f} \leq |f(i)|_i < n_f$$

Define $h \in \Pi K_i$ by

$$h(i) = \begin{cases} \frac{1}{f(i)} & \text{if } i \in X \\ 0 & \text{if } i \notin X \end{cases}$$

Subclaim: $h \in \Pi^* K_i$.
Take $n_h = k_f + 1$. Then

$$|h(i)|_i = \begin{cases} |\frac{1}{f(i)}|_i & \text{if } i \in X \\ 0 & \text{if } i \notin X \end{cases} < k_f + 1 = n_h$$

Subclaim: $[h]_{U^*} [f]_{U^*} = [1]_{U^*}$
For all $i \in X$, $h(i)f(i) = 1$, so $|h(i)f(i) - 1|_i = 0$. This means that $hf U^* 1$, as witnessed by $X_{hf,1}^k = X$

Since all elements have inverses, $\Pi^* K_i / U^*$ is a field. †

Claim 9.4.3. *If there is a prime such that $\{i \in I : K_i \text{ is a field of characteristic } p\} \in U^*$, then $\Pi^* K_i / U$ is a field of characteristic p.*

Proof: If $\{i \in I : K_i \text{ is a field of characteristic } p\} \in U$, then for any $f \in \Pi^* K_i$, $(p \cdot f) U^* 0$. Thus, $\Pi^* K_i / U^*$ has characteristic p. †

Claim 9.4.4. *If each K_i is an algebraically closed field, then so is $\Pi^* K_i / U^*$.*

Proof: We use Claim 9.5.1 that $|\cdot|_*$ is an absolute value; this is proved later but independently. Let $x^n + [f_{n-1}]_{U^*}x^{n-1} + \cdots + [f_0]_{U^*}$ be a polynomial over $\Pi^* K_i / U^*$. For each $i \in I$, there is a root of

$$x^n + f_{n-1}(i)x^{n-1} + \cdots + f_0(i) = 0$$

Let $g \in \Pi K_i$ pick one out for each $i \in I$. By Cauchy's Theorem (reproduced below), we know that

$$|g(i)|_i \leq 1 + \max_k |f_k(i)|_i < 1 + \max_k n_{f_k}$$

Thus, $n_g := 1 + \max_k n_{f_k}$ shows that $g \in \Pi^* K_i$. Then we have

$$[g]^n + [f_{n-1}]_{U^*}[g]^{n-1} + \cdots + [f_0]_{U^*} = [i \mapsto g(i)^n + f_{n-1}(i)g(i)^{n-1} + \cdots + f_0(i)]_{U^*} = [0]_{U^*}$$

So $\Pi^* K_i / U^*$ is algebraically closed. $\qquad\qquad\qquad\qquad\qquad\qquad\dagger$

Cauchy's Theorem used in the proof is a well known bound on the roots of a polynomial. It is classically proved for real or complex polynomials, but the proof is robust enough for our context; we reproduce a proof to show this.

Fact 9.4.5 (Cauchy). *If $z \in (K, |\cdot|)$ is a root of $x^n + a_{n-1}x^{n-1} + \ldots a_0 \in K[x]$, then*

$$|z| \leq 1 + \max_{0 \leq i \leq n-1} |a_i|$$

Proof: If $|z| \leq 1$, then obvious. Suppose $|z| > 1$ and z is a root. Setting $N = \max_{0 \leq i \leq n-1} |a_i|$, we have

$$
\begin{aligned}
z^n &= -a_0 - a_1 z - \cdots - a_{n-1} z^{n-1} \\
|z|^n &\leq |a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1} \\
&\leq N(1 + |z| + \cdots + |z|^{n-1}) \\
&\leq N\frac{|z|^n - 1}{|z| - 1} \\
|z| - 1 &\leq N\frac{|z|^n - 1}{|z|^n} \leq N \\
|z| &\leq 1 + N
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\dagger$

9.5   Properties of $|\cdot|_*$

We now explore the properties of $|\cdot|_*$. As above, its properties are inherited from the properties of U-many $|\cdot|_i$.

Claim 9.5.1. *$|\cdot|_*$ is an absolute value.*

Proof:

I. Clearly, $|[0]_{U^*}|_* = \lim_U |0|_i = 0$. Now suppose $|[f]_{U^*}|_* = 0$. By the definition of U-limit, for all $k < \omega$, we have $\{i \in I : |f(i)|_i < \frac{1}{k}\} \in U$. This is exactly the definition of fU^*0, so $[f]_{U^*} = [0]_{U^*}$.

II. Clear, because $\lim_U (a_i b_i) = (\lim_U a_i)(\lim_U b_i)$.

III. Clear, because $\lim_U (a_i + b_i) = \lim_U a_i + \lim_U b_i$ and $a_i \leq b_i$ on a U-large set implies $\lim_U a_i \leq \lim_U b_i$.

$\dagger$

Recall that an absolute value is *non-trivial* iff the value group properly contains $\{0, 1\}$.

Claim 9.5.2. $|\cdot|_*$ *is a nontrivial absolute value iff there is are $m < n < \omega$ such that $\{i \in I : \exists k_i \in K_i . m < |k_i|_i < n\} \in U$.*

Proof: If $|\cdot|_*$ is nontrivial, then there is some $[f]_{U^*}$ such that $|[f]_{U^*}|_* > m > 1$. Then each $f(i)$ witnesses the existential in $\{i \in I : \exists k_i \in K_i . m < |k_i|_i < n_f\} \in U$.

If there are such $m < n < \omega$, let $f \in \Pi K_i$ pick out a witness on a U-large set X and be 0 elsewhere. Then n witnesses that $f \in \Pi^* K_i$ and X witnesses that $|[f]_{U^*}|_* = \lim_U |f(i)|_i > m$. $\dagger$

We have the following natural condition for the following claim to apply.

Corollary 9.5.3. *If each $(K_i, |\cdot|_i)$ is non-trivial and algebraically closed, then $|\cdot|_*$ is non-trivial.*

Proof: We show that the second half of Claim 9.5.2 is true with $m = 2$ and $n = 3$. Since K_i is nontrivial, there is some $k_i \in K_i$ such that $|k_i|_i > 1$. Since K_i is algebraically closed, we can find $r_i \in \mathbb{Q}$ such that $2 < |k_i^{r_i}|_i < 3$. Thus, $[i \mapsto k_i^{r_i}]_{U^*}$ is in $\Pi^* K_i / U^*$ and has absolute value in $[2, 3]$. $\dagger$

The following example shows that the condition in Claim 9.5.2 is not always satisfied.

Example 9.5.4. *There is $(K_i, |\cdot|_i)$ such that each $|\cdot|_i$ is non trivial on K_i, but $|\cdot|_n$ is trivial on $\Pi^* K_i / U^*$.*

Proof: Take $I = \omega$ and set $K_n = (\mathbb{Q}, |\cdot|_2^n)$; that is, the absolute value on the nth copy of $\mathbb{Q}$ is the standard 2-adic absolute value raised to the n. Note that the value group of K_n is $2^{n\mathbb{Z}}$. Then let $[f]_{U^*} \Pi^* K_n / U^*$ such that $|[f]_{U^*}|_* \neq 1$. We claim that $|[f]_{U^*}|_* = 0$. Suppose not. Then by considering $[f]_{U^*}$ and $\frac{1}{[f]_{U^*}}$, we may assume $|[f]_{U^*}|_* > 1$. Then

$$ X := \{n < \omega : |f(n)|_n > 1\} \in U $$

But, by examining the value group, for each $n < \omega$, $|f(n)|_n > 1$ implies $|f(n)|_n > 2^n$. Then $f \neq \Pi^* K_n$, a contradiction. So $|\cdot|_*$ is trivial. $\dagger$

Claim 9.5.5. *If $|\cdot|_i$ is non-Archimedean, then so is $|\cdot|_*$.*

Proof: Easy from the same facts that shows $|\cdot|_*$ is an absolute value. $\dagger$

Claim 9.5.6. *If $(K_i, |\cdot|_i)$ is complete, then so is $(\Pi^* K_i / U^*, |\cdot|_*)$.*

9.6 Examples

Now we compute some examples of some ultraproducts of p-adics. Let U be an ultrafilter on the index set. As a refresher, recall that any $\frac{a}{b} \in \mathbb{Q} - \{0\}$ can be uniquely factored as $\Pi_{p \text{ prime}} p^{e_p}$ with each $e_p \in \mathbb{Z}$. Then the p-adic absolute value $|\frac{a}{b}|_p = p^{-e_p}$. The choice of base is not crucial; any $\gamma > 1$ will yield an equivalent absolute value and, moreover, any absolute value on $\mathbb{Q}$ is equivalent to the standard one or some p-adic absolute value. Let $\mathbb{Q}_p$ denote $(\mathbb{Q}, |\cdot|_p)$.

1. Fix a prime p. We compute $\Pi^* \mathbb{Q}_p / U^*$ when U is an ultrafilter on ω.

   a) The universe of $\Pi^* \mathbb{Q}_p / U^*$ is the subset of $\Pi \mathbb{Q} / U$ given by

   $$\frac{a}{b} \in \Pi^* \mathbb{Q}_p / U^* \iff \frac{a}{b} \in \Pi \mathbb{Q} / U \text{ and there is some } n < \omega \text{ such that } p^{n+1} \not| ab \text{ or } \frac{a}{b} = 0$$

   That is, $\frac{a}{b}$ is the ratio of elements that are only divided by p a finite number of times, although it can be divisible by other standard primes or infinite primes arbitrarily many times.

   b) Addition, multiplication, and subtraction are inherited from $\Pi \mathbb{Q} / U$.

   c) $|\frac{a}{b}|_*$ counts the multiplicity of p in the prime factorization of $\frac{a}{b}$ in $\Pi \mathbb{Q} / U$ seen as the field of fractions of $\Pi \mathbb{N} / U$; we know that this is some element of $\mathbb{Z}$.

2. Fix a prime p and an ultrafilter U on ω. For each $n < \omega$, set $K_n = (\mathbb{Q}, |\cdot|_n)$ where $|\frac{a}{b}|_n = p^{-n e_p}$; note that this is equivalent to the p-adic absolute value for each $n < \omega$. Considering $[n \mapsto p]_{U^*}$ as an element of $\Pi^* \hat{K}_n / U^*$, this is equal to $[0]_{U^*}$ since its absolute value is $\lim_{n \to \infty} p^{-n} = 0$. However, considering $[n \mapsto p]_{U^*}$ as an element of $\Pi^* \hat{\mathbb{Q}}_p / U^*$, this is nonzero since it's absolute value is $\lim_{n \to \infty} p^{-1} \neq 0$. Thus, one must be careful when dealing with abstract Archimedean value groups and take the specific isomorphism to a subset of $\mathbb{R}$ into account; there are many. In particular, Theorem 9.7.4 requires that the absolute values of each copy of K are actually equal, rather than just equivalent.

3. Let $\langle p_n : n < \omega \rangle$ enumerate the primes of $\mathbb{N}$ and let U be an ultrafilter on ω. Let $(K_n, |\cdot|_n) = (\mathbb{Q}, |\cdot|_n)$, where $|\frac{a}{b}|_i = 2^{-e_{p_n}}$; that is, the p_n-adic absolute value normalized to 2.

   a) The universe of $\Pi^* \mathbb{Q}_{p_i} / U^*$ is the subset of $\Pi \mathbb{Q} / U$ given by

   $$\frac{a}{b} \in \Pi^* \mathbb{Q}_{p_i} / U^* \iff \frac{a}{b} \in \Pi \mathbb{Q} / U \text{ and there is some } n < \omega \text{ such that } p_i^{n+1} \not| ab \text{ for all } i < \omega \text{ or } \frac{a}{b} = 0$$

   That is, $\frac{a}{b}$ is the ratio of elements that are only divided by each standard prime only a bounded number of times, although they may be divisible by inifinitely many primes and by nonstandard primes arbitrarily.

   b) Addition, multiplication, and subtraction are inherited from $\Pi \mathbb{Q} / U$.

   c) To compute $|\frac{a}{b}|_*$ with $a \neq 0$, first note that, for each $i < \omega$, the normalized p_i-adic absolute value (naturally extended to $\Pi^* \mathbb{Q}_{p_i} / U^*$) is in $2^{[-n,n] \cap \mathbb{Z}}$ for n as in 3a. Thus, setting

   $$X_k := \{ i < \omega : |\frac{a}{b}|_{p_i} = 2^k \}$$

   for $k \in [-n, n] \cap \mathbb{Z}$, the X_k partition ω into finitely many pieces. So, there is $k_0 < \omega$ such that $X_{k_0} \in U$. Then, $|\frac{a}{b}|_* = 2^{k_0}$.

Note that this means that there are $2^{2^{\aleph_0}}$ many inequivalent absolute values on the universe of $\Pi^*\mathbb{Q}/U^*$ described in 3a (note that this does not actually depend on U): the extension of the p-adic for each standard prime p and $|\cdot|_*$ for each ultrafilter U (see Jech [Jec06].7.6; compare this with $\mathbb{Q}$.

Note also that the examples demonstrate the fragility of the construction under expanding the language. In particular, the first example $\Pi^*\mathbb{Q}_p/U^*$ is *not* closed under exponentiation:

The functions $[n \mapsto \frac{1}{p}]_{U^*}$ and $[n \mapsto pn+1]_{U^*}$ are both in $\Pi^*\mathbb{Q}_p/U^*$ because $|\frac{1}{p}|_p = p^{-1}$ and $|pn+1|_p = 1$. However, the natural definition of exponentiation creates an element not in $\Pi^*\mathbb{Q}_p/U^*$: we have $|\frac{1}{p^{pn+1}}|_p = p^{pn+1}$ is unbounded, so $[n \mapsto \frac{1}{p}]_{U^*}^{[n \mapsto pn+1]_{U^*}} = [n \mapsto \frac{1}{p^{pn+1}}]_{U^*}$ is not in $\Pi^*\mathbb{Q}_p/U^*$.

9.7 Model theoretic Properties

We now explore some model-theoretic ideas in this context. In this section, we assume all absolute values are non-Archimedean and nontrivial. First, we can give a semantic interpretation of syntactic types. As above, $\bar{A}$ denotes the algebraic closure of A.

Proposition 9.7.1. *Suppose that $(K_1, |\cdot|_1)$ and $(K_2, |\cdot|_2)$ are algebraically closed fields with absolute values that both contain A and their absolute values agree on A. If $a_\ell \in K_\ell$ for $\ell = 1, 2$ such that*

$$tp(a_1/A; K_1) = tp(a_2/A; K_2)$$

then there is $f : \bar{A}a_1 \cong_A \bar{A}a_2$ such that $f(a_1) = a_2$.

Proof: We build f by induction. We start with f_0 defined as $id_A \cup \{(a_1, a_2)\}$. Since the first-order types are the same, we can extend this to the field closure of Aa_1. Now we must show that we can extend it to the algebraic closure. The following lemma does this and finishes the proof.

Lemma 9.7.2. *Let $g : (K, |\cdot|_K) \cong (L, |\cdot|_L)$, K' be a splitting field for the irreducible polynomial $h(x)$ with an absolute value $|\cdot|_{K'}$ that extends $|\cdot|_K$, and L' be a splitting field for the irreducible polynomial $h^g(x)$ with an absolute value $|\cdot|_{L'}$ that extends $|\cdot|_L$. Then there is $g' : (K', |\cdot|_{K'}) \cong (L', |\cdot|_{L'})$ extending g.*

Proof: By results of Galois theory, we can find a field isomorphism $f : K' \cong L'$ extending g. This gives two non-Archimedean absolute values on L': $|x|_{L'}$ and $|f^{-1}(x)|_{K'}$. They both extend $|x|_L$ and $L' \mid L$ is a normal extension of finite degree. By [Rib99].4.2, there is some $\sigma \in Aut_L L'$ such that $|x|_{L'} = |f^{-1}(\sigma(x))|_{K'}$. Then $g' := \sigma^{-1} \circ f$ is the desired function. †

To make full use of the ultraproduct, we need to connect the behavior of the ultraproduct to the behavior or a U-large set. However, the following example shows that the classic version of Łoś' Theorem fails.

Example 9.7.3. *Take the ultrapower of countably many copies of the 2-adics, ie $\Pi^*\mathbb{Q}_2/U^*$ where U is any ultrafilter on ω. Set $f : \omega \to \mathbb{Q}$ by $f(n) = 2^n$. Then, fU^*0, so $\Pi^*\mathbb{Q}_2/U^* \vDash [f]_{U^*} = [0]_{U^*}$. However, $\{n < \omega : \mathbb{Q}_2 \vDash f(n) = 0\} = \emptyset$.*

The problem here is obvious: the ultraproduct considers things that are approximately equal as equal, even if they are actually equal nowhere. Thus, we prove an approximate version of Łos' Theorem; see Theorem 9.7.5 below. First, we show that, if we just consider the constant functions in an ultrapower, we already have a result.

Theorem 9.7.4. *Let $(K, |\cdot|_K)$ be an algebraically closed field with non-Archimedean absolute value and let U be an ultrafilter on I. Then the ultrapower map $j : K \to \Pi^* K/U^*$ defined by $j(k) = [i \mapsto k]_{U^*}$ is elementary.*

Proof: We show that $j(K) \subset \Pi^* K/U^*$; by Robinson [Rob56].3.4.21, ACVF is model-complete so this is sufficient. We show this by induction on terms:

- For $\diamond$ being $+, -$, or $\cdot$ and $k_0, k_1 \in K$, we have

$$
\begin{aligned}
j(k_0 \diamond k_1) &= [i \mapsto k_0 \diamond k_1]_{U^*} = [i \mapsto k_0]_{U^*} \diamond [i \mapsto k_1]_{U^*} \\
&= j(k_0) \diamond j(k_1)
\end{aligned}
$$

-

$$
|j(k)|_* = \lim_U |j(k)(i)|_i = \lim_U |k| = |k|
$$

Thus, $j : K \prec \Pi^* K/U^*$. †

In this proof, we made essential use of the model theoretic properties of ACVF. The same is true of the approximate version of Łos' Theorem. Robinson showed in [Rob56] that ACVF has quantifier elimination by abstract methods. It follows that the K-definable subsets of K^n are Boolean combinations of $\{\mathbf{x} : f(\mathbf{x}) = 0\}$ and $\{\mathbf{x} : |f(\mathbf{x})| \geq |g(\mathbf{x})|\}$ for polynomials $f, g \in K[x]^n$; see Holly [Hol95]. That is, any $\phi(\mathbf{x})$ is equivalent to

$$
\bigvee_{l < n_\phi} [(\bigwedge_{j < n_\phi^{0,l}} |f_{\phi,j}^{0,l}(\mathbf{x})| = 0) \wedge (\bigwedge_{j < n_\phi^{1,l}} |f_{\phi,j}^{1,l}(\mathbf{x})| \neq 0) \wedge (\bigwedge_{j < n_\phi^{2,l}} |f_{\phi,j}^{2,l}(\mathbf{x})| \geq |g_{\phi,j}^{2,l}(\mathbf{x})|)]
$$

for $f_{\phi,j}^{k,l}, g_{\phi,j}^{k,l} \in K[\mathbf{x}]$ for the appropriate indices. Then, given $\phi(\mathbf{x})$ and $k < \omega$, we define an approximation $\phi_k^*(\mathbf{x})$ by

$$
\bigvee_{l < n_\phi} [(\bigwedge_{j < n_\phi^{0,l}} |f_{\phi,j}^{0,l}(\mathbf{x})| < \frac{1}{k}) \wedge (\bigwedge_{j < n_\phi^{1,l}} |f_{\phi,j}^{1,l}(\mathbf{x})| > \frac{1}{k}) \wedge (\bigwedge_{j < n_\phi^{2,l}} |f_{\phi,j}^{2,l}(\mathbf{x})| \geq |g_{\phi,j}^{2,l}(\mathbf{x})| - \frac{1}{k})]
$$

There is some inexactness in this definition, but it is not crucial: $\phi(\mathbf{x})$ might be equivalent to several different Boolean combinations of that form, which would give rise to different computations of ϕ_k^*. However, it suffices to pick one of these forms. The only relation between them that we use is that

$$
(\vee_{j<n} \wedge_{l<m_j} \psi_{j,l})_k^* \equiv \vee_{j<n} \wedge_{l<m_j} (\psi_{j,l})_k^*
$$

when each $\psi_{j,l}$ is one of the 'basic' forms : $|f(\mathbf{x})| = 0$, $|f(\mathbf{x})| \neq 0$, or $|f(\mathbf{x})| \geq |g(\mathbf{x})|$.

In what follows, "$\exists^{cof} k < \omega.\psi$" is an abbreviation of "$\exists k_0 < \omega.\forall k < \omega(k > k_0 \implies \psi)$" and "$\exists^\infty k < \omega.\psi$" is an abbreviation for "$\forall k' < \omega, \exists k > k'.\psi$." In particular, "$cof$" stands for "cofinitely."

Theorem 9.7.5 (Approximate Łos' Theorem). *Let $(K_i, |\cdot|_i)$ be an algebraically closed field with absolute value for $i \in I$ and let U be an ultrafilter on I. If $\phi(\mathbf{x}) \in L_{val}$ such that all free variables are from the field sort and $[h_0]_{U^*}, \ldots, [h_n]_{U^*} \in \Pi^* K_i / U^*$, then the following are equivalent:*

(A) $(\Pi^* K_i / U^*, |\cdot|_*) \vDash \phi([h_0]_{U^*}, \ldots, [h_n]_{U^*})$

(B) $\exists^{cof} k < \omega$ st $\{i \in I : (K_i, |\cdot|_i) \vDash \phi_k^*(h_0(i), \ldots, h_n(i))\} \in U$

(C) $\exists^{\infty} k < \omega$ st $\{i \in I : (K_i, |\cdot|_i) \vDash \phi_k^*(h_0(i), \ldots, h_n(i))\} \in U$

Proof: First, we prove this for the 'basic' terms: $|f(\mathbf{x})| = 0$, $|f(\mathbf{x})| \neq 0$, and $|f(\mathbf{x})| \geq |g(\mathbf{x})|$. We break into cases based on which term it is.

- $\underline{\phi(\mathbf{x}) \equiv \text{``} |f(\mathbf{x})| = 0 \text{''}}$
  Then $\phi_k^*(\mathbf{x}) \equiv \text{``} |f(\mathbf{x})| < \frac{1}{k} \text{.''}$ By definition,

$$\Pi^* K_i / U^* \vDash |f([h_0]_{U^*}, \ldots, [h_n]_{U^*})| = 0 \iff \lim_U |f(h_0(i), \ldots, h_n(i))|_i = 0$$

$$\iff \forall k < \omega, \{i \in I : |f(h_0(i), \ldots, h_n(i))|_i < \frac{1}{k}\} \in U$$

$$\iff \exists^{cof} k < \omega, \{i \in I : |f(h_0(i), \ldots, h_n(i))|_i < \frac{1}{k}\} \in U$$

$$\iff \exists^{\infty} k < \omega, \{i \in I : |f(h_0(i), \ldots, h_n(i))|_i < \frac{1}{k}\} \in U$$

  Where the equivalence of $\forall k$; $\exists^{cof} k$; and $\exists^{\infty} k$ is because, for the statements following, being true for k implies it is true for $k - 1$.

- $\underline{\phi(\mathbf{x}) \equiv \text{``} |f(\mathbf{x})| \neq 0 \text{''}}$
  Then $\phi_k^*(\mathbf{x}) \equiv \text{``} |f(\mathbf{x})| > \frac{1}{k} \text{.''}$ By definition,

$$\Pi^* K_i / U^* \vDash |f([h_0]_{U^*}, \quad \ldots \quad , [h_n]_{U^*})| \neq 0$$

$$\iff \exists k_0 < \omega. \lim_U |f(h_0(i), \ldots, h_n(i))|_i > \frac{1}{k_0}$$

$$\iff \exists k_0 < \omega. \forall k > k_0, \lim_U |f(h_0(i), \ldots, h_n(i))|_i > \frac{1}{k}$$

$$\iff \exists k_0 < \omega. \forall k > k_0, \{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > \frac{1}{k}\} \in U$$

$$\iff \exists^{cof} k < \omega. \{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > \frac{1}{k}\} \in U$$

$$\iff \exists^{\infty} k < \omega. \{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > \frac{1}{k}\} \in U$$

  Where the equivalence of $\exists^{cof} k$ and $\exists^{\infty} k$ is because, for the statements following, being true for k implies it is true for $k + 1$.

- $\phi(\mathbf{x}) \equiv \text{``}|f(\mathbf{x})| \geq |g(\mathbf{x})|\text{''}$

  Then $\phi_k^*(\mathbf{x}) \equiv \text{``}|f(\mathbf{x}) \geq |g(\mathbf{x})| - \frac{1}{k}.\text{''}$ First, suppose

$$
\begin{aligned}
\Pi^* K_i / U^* \quad &\vDash \quad |f([h_0]_{U^*}, \ldots, [h_n]_{U^*})| \geq |g([h_0]_{U^*}, \ldots, [h_n]_{U^*})| \\
&\implies \quad \forall k < \omega, |f([h_0]_{U^*}, \ldots, [h_n]_{U^*})| > |g([h_0]_{U^*}, \ldots, [h_n]_{U^*})| - \frac{1}{k} \\
&\implies \quad \forall k < \omega, \exists c_k \in \mathbb{R}.|f([h_0]_{U^*}, \ldots, [h_n]_{U^*})| > c_k \text{ and} \\
&\qquad c_k + \frac{1}{k} > |g([h_0]_{U^*}, \ldots, [h_n]_{U^*})| \\
&\implies \quad \forall k < \omega, \exists c_k \in \mathbb{R}.\{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > c_k \text{ and} \\
&\qquad c_k + \frac{1}{k} > |g(h_0(i), \ldots, h_n(i))|\} \in U \\
&\implies \quad \forall k < \omega, \{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > |g(h_0(i), \ldots, h_n(i))| - \frac{1}{k}\} \in U \\
&\implies \quad \exists^{cof} k < \omega.\{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > |g(h_0(i), \ldots, h_n(i))| - \frac{1}{k}\} \in U \\
&\implies \quad \exists^{\infty} k < \omega.\{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > |g(h_0(i), \ldots, h_n(i))| - \frac{1}{k}\} \in U
\end{aligned}
$$

  Now suppose that $k < \omega$ such that

$$
\{i \in I : |f(h_0(i), \ldots, h_n(i))|_i > |g(h_0(i), \ldots, h_n(i))|_i - \frac{1}{k}\} \in U
$$

  By the monotonicity properties of the U-limit,

$$
\begin{aligned}
|f([h_0]_{U^*}, \ldots, [h_n]_{U^*})|_* &= \lim_U |f(h_0(i), \ldots, h_n(i))|_i \\
&\geq \lim_U |g(h_0(i), \ldots, h_n(i))|_i - \frac{1}{k} = |g([h_0]_{U^*}, \ldots, [h_n]_{U^*})|_* - \frac{1}{k}
\end{aligned}
$$

  Taking the supremum over cofinally (or just infinitely) many $k < \omega$, we get the desired result.

Now we suppose that $\phi \equiv \vee_{j<n} \wedge_{l<m_j} \psi_{j,l}$ where $\psi_{j,l}$ are of the above form; we suppress the parameters for notational ease. By the above quantifier elimination, this covers all formulas. For additional ease, let K^* denote $\Pi^* K_i / U^*$. By the above, we already know that

$$
\begin{aligned}
K^* \vDash \psi_{j,l} \quad &\Longleftrightarrow \quad \exists^{cof} k \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U \\
&\Longleftrightarrow \quad \exists^{\infty} k \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U
\end{aligned}
$$

Now we have the following; numbered equations have additional explanations later (and unnumbered equations are deemed to be 'obvious').

(A) $\implies$ (B)

$$
\begin{aligned}
K^* \vDash \phi \quad &\implies \quad \exists j < n \forall l < m_j K^* \vDash \psi_{j,l} \\
&\implies \quad \exists j < n \forall l < m_j \exists^{cof} k \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U \\
&\implies \quad \exists j < n \exists^{cof} k \forall l < m_j \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U \qquad (9.1) \\
&\implies \quad \exists^{cof} k \exists j < n \forall l < m_j \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U \qquad (9.2) \\
&\implies \quad \exists^{cof} k \{i \in I : K_i \vDash \vee_{j<n} \wedge_{l<m_j} (\psi_{j,l})_k^*\} \in U \\
&\implies \quad \exists^{cof} k \{i \in I : K_i \vDash (\phi)_k^*\} \in U \qquad (9.3)
\end{aligned}
$$

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For implication 9.1, we can take the maximum of all thresholds for each l; this will be a threshold for all $l < m_j$ as desired. For implication 9.2, note that $\exists^{cof}$ is short for a $\exists\forall$-formula and, in general, $\exists(\exists\forall)$ implies $(\exists\forall)\exists$. For implication 9.3, note that $\phi_k^* \equiv \vee_{j<n} \wedge_{l<m_j} (\psi_{j,l})_k^*$.

(B) $\implies$ (C) Clear because $\exists^{cof}$ implies $\exists^\infty$.

(C) $\implies$ (A)

$$\begin{aligned}
\exists^\infty k\{i \in K_i \vDash \phi_k^*\} \in U \quad &\implies \quad \exists^\infty k \exists j < n \forall l < m_j \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U \\
&\implies \quad \exists j < n \exists^\infty k \forall l < m_j \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U \quad (9.4) \\
&\implies \quad \exists j < n \forall l < m_j \exists^\infty k \{i \in I : K_i \vDash (\psi_{j,l})_k^*\} \in U \quad (9.5) \\
&\implies \quad \exists j < n \forall l < m_j K^* \vDash \psi_{j,l} \\
&\implies \quad K^* \vDash \phi
\end{aligned}$$

Implication 9.4 holds by the Pigeonhole Principle: each of infinitely many k is associated with some j from a finite set (i.e. n); thus there is some j that has infinitely many k associated with it. For implication 9.5, we use that $\exists^{inf}$ is short for a $\forall\exists$ formula and that, in general $(\forall\exists)\exists$ implies $\exists(\forall\exists)$.†

Note that the above results relied heavily on the field being algebraically closed. A direct proof of the above theorem for non-algebraically closed fields would likely be more difficult as it cannot rely on quantifier elimination. Instead, one can simply close each of the fields algebraically, extend the absolute values as in [Rib99].§4, and then apply the above result.

We conclude with an analysis of the value group of $(\Pi^* K_i/U^*, |\cdot|_*)$; denote this Γ_*. Recall that the standard ultraproduct has as its value group the ultraproduct of the individual value groups. That is, Γ_* consists of the ultralimit of all sequences. In this case, we get that Γ_* is the ultralimit of all *bounded* sequences

Proposition 9.7.6. Γ_* *is the U-limit of all bounded sequences from $\Pi\Gamma_i$.*

Proof: This is by the definition. First, suppose γ is such a U-limit, say of $\{\gamma_i : i \in I\}$. Since the absolute value is surjective, there are $a_i \in K_i$ such that $|a_i|_i = \gamma_i$. Then $[i \mapsto a_i]_{U^*} \in \Pi^* K_i/U^*$ by the boundednes of $\{\gamma_i : i \in I\}$ and it has absolute value γ. On the other hand, suppose $\gamma \in \Gamma_*$ and $[f]_{U^*} \in \Pi^* K_i/U^*$ has absolute value γ. Then, $\{|f(i)|_i : i \in I\}$ is a bounded sequences in $\Pi\Gamma_i$ with U-limit γ. †

Two special cases bear mentioning. Suppose the value group of each $(K_i, |\cdot|_i)$ is Γ. Then the above proposition implies that $\Gamma_* = \bar{\Gamma}$, the closure of Γ in $\mathbb{R}$. Moreover, if Γ is discrete, then $\Gamma_* = \Gamma$.

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