A Small Observation on Co-categories

Peter LeFanu Lumsdaine

plumsdai@andrew.cmu.edu

19 June, 2008

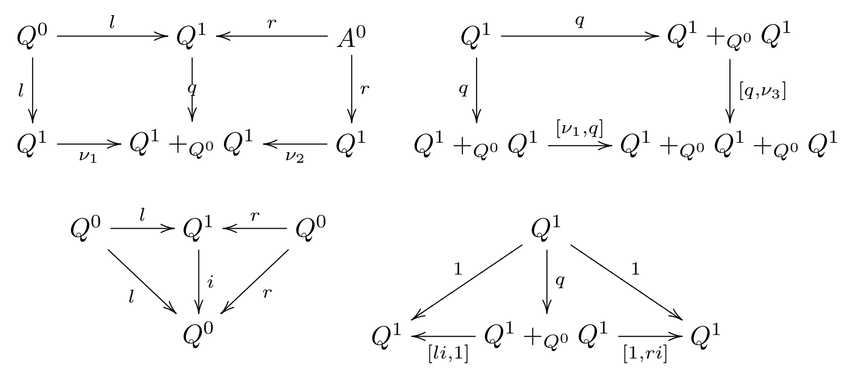
Abstract

Various concerns suggest looking for internal co-categories in categories with strong logical structure. It turns out that in any coherent category $\mathcal{E}$, all co-categories are co-equivalence relations.

Definition 1 *Let $\mathcal{E}$ be any category. An (internal)* co-category $\mathbf{Q}$ *in $\mathcal{E}$ is an internal category in $\mathcal{E}^{op}$, i.e. objects and morphisms in $\mathcal{E}$*

$$Q^0 \underset{r}{\overset{l}{\underset{\xleftarrow{\;i\;}}{\rightrightarrows}}} Q^1 \xrightarrow{q} Q^1 +_{Q^0} Q^1$$

such that the following diagrams commute:



Definition 2 *A co-category $\mathbf{Q}$ is a* co-preorder *if the maps l, r are jointly epimorphic.*

A co-category $\mathbf{Q}$ is a co-groupoid *if there is a map $s : Q^1 \to Q^1$ satisfying the duals of the usual identities for the inverse map of a groupoid.*

A co-groupoid $\mathbf{Q}$ is a co-equivalence relation *if it is a co-preorder.*

Remark 1 *In a co-preorder, the co-composition q is uniquely determined by the maps l, r, i; likewise, in a co-groupoid, the co-inverse map s is determined by the rest of the structure.*

Together with the obvious notion of morphism of co-categories, these give categories and full inclusions

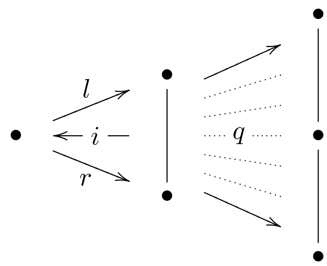
$$\mathbf{CoEqRel}(\mathcal{E}) \hookrightarrow \mathbf{CoPreOrd}(\mathcal{E}) \hookrightarrow \mathbf{CoCat}(\mathcal{E}).$$

Example If $\mathcal{E}$ has all (or enough) pushouts and $m : S \rightarrowtail A$ is any monomorphism, then the *co-kernel pair* of m is a co-equivalence relation

$$A \underset{\nu_2}{\overset{\nu_1}{\rightrightarrows}} \xleftarrow{[1_A, 1_A]} A +_S A \xrightarrow{[\nu_1, \nu_3]} A +_S A +_S A.$$

This gives the object part of a functor $\mathbf{Mon}(\mathcal{E}) \to \mathbf{CoEqRel}(\mathcal{E})$, which (almost by definition) is one half of an equivalence whenever $\mathcal{E}$ is co-exact.

Example A paradigmatic example is the interval $\mathbf{I}$ in $\mathbf{Top}$, where I^0 is a singleton, I^1 is the unit interval, l and r are the endpoints, $I^1 +_{I^0} I^1$ is two copies of the interval joined end to end, and q is the obvious "stretching" map. Unfortunately, this is also of course not an actual co-category — the axioms hold only up to homotopy. However, it provides a very useful mental picture for the arguments below; and if we delete the interior of the interval, we obtain a genuine co-category. See also the examples below for more versions of the interval.



Definition 3 *A* coherent category *is a category with all finite limits, and images and unions that are stable under pullback.*

[4, A1.3–4] gives various basic results on coherent categories, which we will use here without comment.

Definition 4 Coherent logic *is the fragment of first-order logic built up from atomic formulæ using finite con-/dis-junction and existential quantification.*

Coherent logic is discussed in [4, D1.1–2]; the essential point is that coherent logic can be interpreted soundly in coherent categories, and so may be used as an internal language for working in them.

Proposition 1 *In a coherent category $\mathcal{E}$, every co-category $\mathbf{Q}$ is a co-equivalence relation.*

Proof First, we show that any co-category $\mathbf{Q}$ is a co-preorder.

Arguing in the internal logic: given x in Q^1, consider $q(x)$, in $Q^1 +_{Q_0} Q^1$. Either there is some y in Q^1 with $q(x) = \nu_1(y)$, or else some y with $q(x) = \nu_2(y)$. In the first case, we then have $x = [li, 1]q(x) = li(y)$; in the second, $x = ri(y)$. Thus any x in Q^1 is in the image of either l or r, i.e. l and r are jointly covering, hence epi. (Indeed, in the first case $x = li(y) = l(il)i(y) = li(li(y)) = li(x)$, and in the second, $x = ri(x)$.)

Restating this diagrammatically: $Q^1 +_{Q^0} Q^1$ is the union of the subobjects $\nu_j : Q^1 \to Q^1 +_{Q^0} Q^1$, so Q^1 is the union of the subobjects $m_j = q^*(\nu_j)$:

$$\begin{array}{ccc} P_j & \xrightarrow{q_j} & Q^1 \\ \downarrow{\scriptstyle m_j} & & \downarrow{\scriptstyle \nu_j} \\ Q^1 & \xrightarrow{q} & Q^1 +_{Q^0} Q^1 \end{array}$$

In particular, the pair m_1, m_2 are jointly covering. But by the co-unit identities,

$$liq_1 = [li, 1]\nu_1 q_1 = [li, 1]qm_1 = m_1$$

and $riq_2 = m_2$. Thus liq_1, riq_2 are jointly covering, and hence so are l, r.

Now, we check that any co-preorder is a co-equivalence relation. (We give only the diagrammatic version. Exercise: restate this in the internal logic!) We want to define $s : Q^1 \to Q^1$ with $sl = r$, $sr = l$. Since l, r are monos with union Q^1, the pullback square

$$\begin{array}{ccc} \bullet & \xrightarrow{\pi_2} & Q^0 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle r} \\ Q^0 & \xrightarrow{l} & Q^1 \end{array}$$

is also a pushout, so to construct s as above, it is enough to show that $r\pi_1 = l\pi_2$. But $\pi_1 = il\pi_1 = ir\pi_2 = \pi_2$, so $r\pi_1 = r\pi_2 = l\pi_1 = l\pi_2$, and we are done. □

Corollary 2 *If $\mathcal{E}$ is any coherent category with co-kernel pairs of monos, then* $\boldsymbol{CoCat}(\mathcal{E}) \simeq \boldsymbol{Mon}(\mathcal{E})$.

Proof A coherent category is certainly co-effective, so if it has co-kernel pairs, it is co-exact. □

Corollary 3 *For any topos $\mathcal{E}$,* $\boldsymbol{CoCat}(\mathcal{E}) \simeq \mathcal{E}/\Omega$. □

In particular, inspecting this equivalence, we see that in this case there is a universal internal co-category in $\mathcal{E}$, from which every co-category in $\mathcal{E}$ may be obtained uniquely by pullback: it is the co-kernel pair of $\top : 1 \to \Omega$.

Example The condition that unions are preserved by pullback is crucial: **AbGp**, for instance, is regular, and has unions, but there is a non-co-preorder co-category corresponding to the interval pictured above, given by the objects

$$Q^0 = \langle v_0 \rangle \qquad Q^1 = \langle v_0, e_1, v_1 \rangle \qquad Q^1 +_{Q^0} Q^1 = \langle v_0, e_1, v_1, e_2, v_2 \rangle$$

(with the natural maps making this a pushout), and maps given by the matrices

$$l = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad i = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \quad q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This example may be given more structure; it is, for instance, the total space of an natural co-category in **Ch**(**AbGp**). Since all the underlying groups are free and of finite rank, dualising by transposing matrices also gives corresponding categories in **AbGp** and **Ch**(**AbGp**).

However, any category in a Mal'cev category is a groupoid (this has been observed by various authors, e.g. in [3]), so any co-category in a co-Mal'cev category (e.g. in an Abelian category, or a topos [2]) is a co-groupoid.

Example An example of a non-co-groupoid co-category is the interval **I** in **Cat**, with $I^0 = (\cdot)$, $I^1 = (\cdot \to \cdot)$; seen as a co-simplicial object, this is just the usual inclusion functor $\Delta \hookrightarrow$ **PreOrd** $\hookrightarrow$ **Cat**.

Indeed, the functor **Cat** $\to$ **SSet** $\to$ **SAbGp** $\to$ **Ch**(**AbGp**) $\to$ **Ch**(**AbGp**) "take nerve; take free abelian groups; normalise to a complex; quotient out by subcomplex generated in degrees ≥ 2" sends **I** to the co-category in **Ch**(**AbGp**) of the previous example.

Co-categories arise as candidate "interval objects" when using 2-categories to model intensional type theory [1]. There, one seeks them in categories with some sort of "weakened" logical structure; the present result confirms the suspicion that examples in classical "strict" logical categories are necessarily fairly trivial.

Many thanks are due to Steve Awodey, for originally posing the question of what co-categories could exist in a topos, and Peter Johnstone, for suggesting and improving parts of the proofs.

References

[1] Steve Awodey and Michael A. Warren, *Homotopy-theoretic models of identity types*, Math. Proc. Cambridge Philos. Soc. 146 (2009), no.1, pp.45–55, `arXiv:0709.0248 [math.LO]`

[2] Dominique Bourn, *Mal'cev Categories and Fibration of Pointed Obects*, Applied Categorical Structures, Vol.4 (1996), pp.307–327

[3] A. Carboni, G.M. Kelly, M.C. Pedicchio *Some remarks on Maltsev and Goursat categories*, Applied Categorical Structures, Vol.1 (1993), pp.385–421

[4] Peter Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, Oxford University Press (2002)