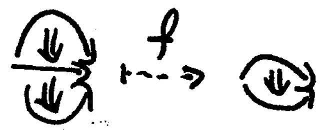
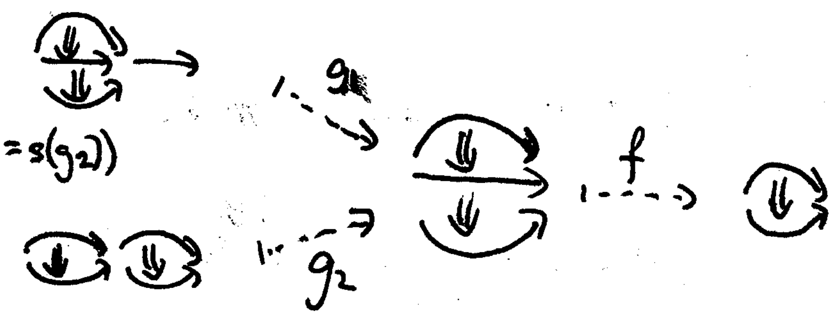
→ Recall: a globular operad is: sets $P(\pi)$ of "$\pi$-ary operations" ("symbols") for each pasting diag. $\pi$, with source & target maps $s : P(\pi) \to P(s(\pi))$, $t : \ldots$,
draw ~~as~~ $f \in P(\cdot)$ as



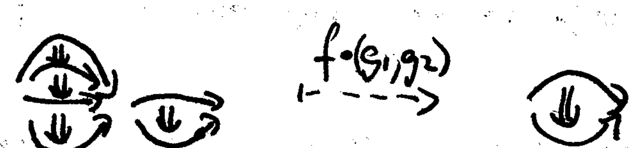

;

~~now have~~ ... also, composition maps, giving eg
for

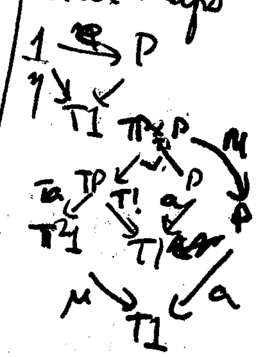



(with $t(g_1) = s(g_2)$)

a composite




A might want to describe explicitly in these terms, somewhat economically, a glob. set $P \xrightarrow{a} T1$ with maps




→ Want to be able to define a weak ω-category as an alg. for ~~the~~ such an operad. (All operads today: globular).

Not just any operad: ~~two~~ essential problems:

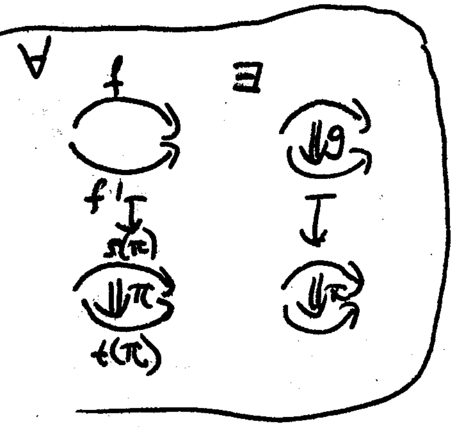
→ desired composition laws might not exist!

→ or might have multiple composition laws ~~for each~~ for a diag.

(Failure of associativity.) We want all comp'n laws for a diag. to agree "up to homotopy", i.e. up to comp'n laws giving cells of the next dim'n, & these should be coherent, etc...

→ Solve both problems at once: CONTRACTIBILITY!

**Defn:**
→ A glob operad $P \xrightarrow{a} T1$ is a map $\chi$
contraction on a




giving, for ~~each~~ any $\pi \in T1(n)$
& any parallel pair $f, f' \in P(n-1)$
with $a(f) = s(\pi)$, $a(f') = t(\pi)$,
a lifting $g = \chi(f, f', \pi) \in P(n)$ with $f \xrightarrow{g} f'$ (i.e. $s(g) = f$, $t(g) = f'$)
$a(g) = \pi$.

(No other axioms on $\chi$.)

→ This ensures we do have comp'n laws: e.g. get an elt $g \in P(\cdot \to \cdot \to \cdot)$
by lifting $\pi = (\cdot \to \cdot \to \cdot)$ btn the pair $f = f' = e(1) \in P(\cdot)$.

→ Also ensures coherence, assoc: e.g. given $g \in P(\cdot \to \cdot \to \cdot)$, construct
the two associations $f, f' \in P(\cdot \to \to \to \cdot_1)$ (using composition map)
Get $h : f \to f'$ in $P(\cdot \to \cdot \to \cdot \to \cdot_2)$
since $\cdot \to \cdot \to \cdot \to \cdot_2$ is a 2-cell from $\cdot \to \cdot \to \cdot \to \cdot$ to itself in $T1$.

Digress: explain degenerate p.d.s, i.e. $\cdot \to \cdot \in T1(1)$, also $\cdot \to_2 \cdot \in T1(2)$ etc. from ids in a strict ω-cat

→ **Theorem:** there is an initial operad-with-contraction, $L$.

Initial in cat where maps must preserve the contr'n

→ **Defn** (Leinster 2004): A weak ω-cat. is
an $L$-algebra, i.e. $X \in \hat{\mathbb{G}}$ with operad map $L \to End(X)$.

→ **Note**: by def'n of $L$, to give wk-ω-cat strux on $X$, it
suffices to give action of some contractible operad on $X$.

HALLELUJAH! At last! A defn!

→ Back to type theory. (Will recall things as we need them.)

A any (dependent) type ~~in context~~. Then let $\underline{A}_n$ be the (dep't) context

$$x_0, y_0 : A,\ x_1, y_1 : \mathrm{Id}(x_0, y_0),\ \dots\ z_n : \mathrm{Id}(x_{n-1}, y_{n-1}).$$

Recall: for contexts $\Delta$, $\Delta'$, where $\Delta' = x_0 : B_0,\ x_1 : B_1(x_0),\ \dots,\ x_n : B_n(x_0, \dots, x_{n-1})$,

a context map $\sigma : \Delta \to \Delta'$ is a seq. of terms $\sigma_0, \dots, \sigma_n$

s.t. $\Delta \vdash \sigma_0 : B_0$, $\Delta \vdash \sigma_1 : B_1(\sigma_0)$, etc.

→ Theorem: $T$ any extension of $ML_I$, $\Gamma$ any closed context of $T$, $\Gamma \vdash A$ type. Then for any dep't cxt $\Delta$ over $\Gamma$, the glob. set of context maps $\underbrace{\Delta \xrightarrow{\bar{\sigma}} \underline{A}_n}_{\text{"\{terms of type A \& its higher identities\}"}}$ carries a canonical weak $\omega$ cat. structure.

→ Pf: we will give an operad $P_{ML}$ (not depending on $T, \Gamma, A$), show it acts on these globular sets, & show it is contractible.

→ Recall: Any type th'y $T$ (with at least the structural rules of ML) has a syntactic category $\mathcal{C}(T)$, with objects closed contexts of $T$, maps: context maps as above.

Take $T = ML_I[X]$, i.e. $ML_I$ plus a new "generic" base type $X$, no term formation rules for it.

For a p.d. $\pi$, define $\Gamma_\pi$ by: well, e.g. $\pi =$ ,

$$\Gamma_\pi = x,y,z : X,\ f,f' : \mathrm{Id}(x,y),\ g,g' : \mathrm{Id}(y,z),\ p : \mathrm{Id}(f,f'),\ q : \mathrm{Id}(g,g').$$

Generally: use the fact there's a canonical ordering on the $n$-cells of any p.d.

<u>There</u> are just ~~what~~ the limits we need to define ~~the~~ the endo operad of the ~~contexts~~ globular object given by the contexts $\underline{X}_n$ in $\mathcal{C}(\mathrm{ML}_I[X])$

<u>Def'n</u> $P_{ML}$ is this endo. operad.

<u>Now</u>, for any ext'n $T$ of $\mathrm{ML}_I$, any closed type $A$ of $T$, have translation $\mathrm{ML}_I[X] \longrightarrow T$ taking $X$ to $A$, inducing functor $\mathcal{C}(\mathrm{ML}_I[X]) \longrightarrow \mathcal{C}(T)$ taking $\underline{X}_n$ to $\underline{A}_n$, preserving enough limits to give map of operads $\underbrace{\mathrm{End}(\underline{X})}_{= P_{ML}} \longrightarrow \mathrm{End}(\underline{A})$

i.e. ~~…~~ giving action of $P_{ML}$ on $\underline{A}$.

and hence on the "globular hom-sets" $\mathcal{C}(T)[\Delta, \underline{A}]$ as desired.

(Adapt for dependent types $A$ over $\Gamma$ by using "slice type th'y" $T/\Gamma$.)

<u>Now</u>, the heart of the matter: <u>contractibility</u> of $P_{ML}$.

Unwind def'n of ~~glob.~~ endo. operad:

<u>what</u> is an elt $\vec{p}$ of $P_{ML}($ $)$ ??

→ It turns out to be terms $\sigma_0, \tau_0, \sigma_1, \tau_1, \rho$, with

$$
(*)\;\left\{
\begin{array}{l}
x : X \vdash \sigma_0(x) : X \\
z : X \vdash \tau_0(z) : X \\
x, y, z : X,\ f : \mathrm{Id}(x,y),\ g : \mathrm{Id}(y,z) \vdash \sigma_1(x,y,z,f,g) : \mathrm{Id}(\sigma_0(x), \sigma_1(z)) \\
x, y, z : \ldots,\ f' : \ldots,\ g' : \ldots \vdash \tau_1(x,y,z,f',g') : \mathrm{Id}(\sigma_0(x), \sigma_1(z)) \\
x, y, z,\ f, f',\ g, g',\ p, q \vdash \rho(\ldots) : \mathrm{Id}(\sigma_1, \tau_1)
\end{array}
\right.
$$

<u>Idea</u>: given
$$
\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow p"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow q"] & z
\end{tikzcd}
$$
, there are terms

$\sigma_0, \tau_0$ for composing the 0-dim'l source & tgt,

$\sigma_1, \tau_1$ for the 1-d src & tgt,

$\rho$ for the whole thing.

→ <u>Contraction</u>? Need to show: for any $\pi \in \Pi_n$, $\vec{\rho}' \in P(s(\pi))$, $\vec{\rho}'' \in P(t(\pi))$, with $\vec{\rho}', \vec{\rho}''$ parallel, need <u>filler</u> $\vec{\rho} \in P(\pi)$, $s(\vec{\rho}) = \vec{\rho}'$, $t(\vec{\rho}) = \vec{\rho}''$.

<u>Now</u>, $\vec{\rho}'$ is a seq. of terms $\sigma_0', \tau_1', \ldots \sigma_{n-2}', \tau_{n-2}', \rho'$ as in (*) for $s(\pi)$

$\vec{\rho}''$ is ‒‒ ‒‒ ‒‒ $\sigma_0'', \tau_1'' \ldots, \tau_{n-2}'', \rho''$ ‒‒,

saying these are parallel is saying $\sigma_i' = \sigma_i''$, $\tau_i' = \tau_i''$,

<u>so</u> ~~taking~~ setting $\sigma_i := \sigma_i'$ $\tau_i := \tau_i'$ $(i < n-1)$, $\sigma_n = \rho'$, $\tau_n = \rho''$,

we see: a parallel pair needing a filler is <u>exactly</u> a seq. of terms $\sigma_0, \tau_0, \ldots, \sigma_{n-1}, \tau_{n-1}$ as in (*) for $\pi$, & a <u>filler</u> for this is <u>exactly</u> just a term $\rho$ to complete the seq $((\sigma_i, \tau_i), \rho)$ of (*).

<u>So</u>: have, eg terms $x:X \vdash \sigma_0(x):X$, $z:X \vdash \tau_0(z):X$,

$$x,y,z:X,\ f:\mathrm{Id}(x,y),\ g:\mathrm{Id}(y,z) \vdash \sigma_1(x,y,z,f,g):\mathrm{Id}(\sigma_0(x),\tau_0(z))$$

$$x,y,z:X,\ f':\mathrm{Id}(x,y),\ g':\mathrm{Id}(y,z) \vdash \tau_1(x,y,z,f',g'):\mathrm{Id}(\sigma_0(x),\tau_0(z))$$

& need a term

$$x,y,z:X,\ f,f':\mathrm{Id}(x,y),\ g,g':\mathrm{Id}(y,z)\ p:\mathrm{Id}(f,f'),\ q:\mathrm{Id}(g,g') \vdash p:\mathrm{Id}(\sigma_1(x,y,z,f,g),\tau_1(x,y,z,f',g'))$$

So, working bottom-up as usual; we find that by repeatedly using Id-elim, on cells of the highest dimension "first", it's enough to give the filler in the case where $x$ & iterated reflexivity terms $r(x), r(r(x)), \ldots$ have been plugged in for all the variables.

What can we do now? How do we know these now agree?

$$x \vdash ?:\mathrm{Id}(\sigma_1(x,x,x,r(x),r(x)),\tau_1(x,x,x,r(x),r(x)))$$
——————————————————— [Id-elim]
$$x,y,f \vdash ?:\mathrm{Id}(\sigma_1(x,y,y,f,r(y)),\tau_1(x,y,y,f,r(y)))$$
——————————————————— [Id-elim]
$$x,y,z,\ f,g \vdash ?:\mathrm{Id}(\sigma_1(x,y,z,f,g),\tau_1(x,y,z,f,g))$$
——————————————————— [Id-elim]
$$x,y,z,\ f,f',\ g.\ p \vdash ?:\mathrm{Id}(\sigma_1(x,y,z,f,g),\tau_1(x,y,z,f',g))$$
——————————————————— [Id-elim]
$$x,y,z,\ f,f',\ g,g'.\ p,\ q \vdash ?:\mathrm{Id}(\sigma_1(x,y,z,f,g),\tau_1(x,y,z,f',g'))$$

# Normalisation! ← This is the punchline!*

→ Normalisation ensures that any term $x:X \vdash \pi(x):X$ must be ~~of~~ of canonical form, i.e. just equal to $x$ itself.
(equal to one) so $x:X \vdash \sigma_0(x) = \tau_0(x) = x:X$,

next, it ensures that any term $x:X \vdash \pi(x): Id(x,x)$ must be equal to one of canonical form, i.e. to $r(x)$,

so $x:X \vdash \sigma_1(x,x,x,r(x),r(x)) = \tau_1(x,x,x,r(x),r(x)) = r(x): Id(x,x)$.

In the general case, we ~~can~~ show, ~~that~~ by working up thru the dimensions, that after plugging in all iterated $r(-)$ terms, each $\sigma_i, \tau_i$ ~~&~~ must become equal to iterated reflexivity terms $r^i(x)$;

so now we can just take the desired filler to be $\rho(x) := r^n(x)$
$: Id(r^{n-1}(x), r^{n-1}(x))$

so we are done!

This completes the proof of the theorem. $\square$

---

* Actually, I haven't yet checked carefully that existing normalization statements do give this; if they don't, though, I have an alternative proof for this key step, ~~tho~~ by building $P_{ML}$ ~~from~~ in the fragment with just Id-types (no $\Pi$'s, $\Sigma$'s, 1's), & showing (by struct'l ind'n) that the context $x:X$ is initial in this fragment. However, that is much more cumbersome!

→ Obvious questions: we gave wk-ω-cats $\mathcal{C}(T)(\Delta, \underline{A}_\bullet)$, for a type A & ctxt Δ in a t.t. T.

→ This is easily functorial in T & Δ, i.e. translations $T \longrightarrow T'$ & ctxt maps $\Delta' \longrightarrow \Delta$ induce strict maps of wk ω-cats in the obvious way.

→ What about terms $a : A \vdash \tau(a) : B$? These should induce "weak-ω-functors", "weak maps of ω-cats" ~~from~~

$$\tau^* : \mathcal{C}(T)(\Delta, \underline{A}_\bullet) \longrightarrow \mathcal{C}(T)(\Delta, \underline{B}_\bullet)$$

But "weak ω-functor" is a slippery concept, & this doesn't seem simple to show! This is an important question...

Further directions:

→ Can we use this to give a syntactic presentation of wk-ω-gpds? i.e. do all wk-ω-gpds (up to some notion of weak equivalence) arise in this way?

→ We can formalise much maths in extensions of $ML_I$! So, what are the "fund'l wk ω-cats" of various natural types, e.g. Bool, Nat, ...? Their geometric realisations are top. spaces; what do these look like? Are nice descriptions of these attainable at all??