Bisections of graphs

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Abstract

A bisection of a graph is a bipartition of its vertex set in which the number of vertices in the two parts differ by at most 1, and its size is the number of edges which go across the two parts. In this paper, motivated by several questions and conjectures of Bollobás and Scott, we study maximum bisections of graphs. First, we extend the classical Edwards bound on maximum cuts to bisections. A simple corollary of our result implies that every graph on n vertices and m edges with no isolated vertices, and maximum degree at most $n/3+1$, admits a bisection of size at least $m/2+n/6$. Then using the tools that we developed to extend Edwards's bound, we prove a judicious bisection result which states that graphs with large minimum degree have a bisection in which both parts span relatively few edges. A special case of this general theorem answers a conjecture of Bollobás and Scott, and shows that every graph on n vertices and m edges of minimum degree at least 2 admits a bisection in which the number of edges in each part is at most $(1/3+o(1))m$. We also present several other results on bisections of graphs.

1 Introduction

Many classical partitioning problems in Combinatorics and Computer Science seek a partition of a combinatorial object (e.g., a graph, directed graph, hypergraph, etc.) which optimizes a single parameter. Two famous examples are Max Cut and Max Bisection, which have been widely studied in Combinatorics and Computer Science (see, e.g., [13, 14, 17, 26] and their references). In these problems, the objective is to find a bipartition which maximizes the number of crossing edges, with Max Bisection having the additional constraint that the two vertex classes should differ in size by at most one.

The most basic result in that area is that every graph with m edges has a cut of size at least $\frac{m}{2}$. This can be seen by considering the expected number of crossing edges in a uniformly random partition, or by analyzing the following natural greedy algorithm. In this algorithm, one processes the vertices in an arbitrary order, adding each new vertex to the side which maximizes the number of edges crossing back to the previously positioned vertices. Moreover, one can similarly obtain the same estimate for maximum bisection.

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There have been many improvements of the above bound, most notable among which is the following result of Edwards [9] which is tight, e.g., for complete graphs. (Here and in the remainder, $e(U, W)$ denotes the number of edges between two disjoint sets U and W.)

$$\max_{V_1 \cap V_2 = \emptyset} e(V_1, V_2) \ge \left\lceil \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8} \right\rceil . \tag{1}$$

The area of *judicious partitioning* considers the general family of problems in which one seeks to optimize multiple parameters simultaneously. In the Max Cut setting, the canonical example is the beautiful result of Bollobás and Scott which shows that one can find a cut $V_1 \cup V_2$ which not only achieves the Edwards bound (1), but also induces few edges in each V_i.

Theorem 1.1. *(Bollobás and Scott, see [4].) Every graph $G = (V, E)$ with m edges has a bipartition $V = V_1 \cup V_2$ which achieves the bound in (1), and each vertex class spans at most*

$$\frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16}$$

edges.

Note that this simultaneously optimizes three parameters: the number of edges crossing the cut, and the number of edges inside each V_i for $i = 1, 2$. It is impossible to obtain results of this nature through probabilistic arguments which are based on first-moment (expected value) calculations alone. Indeed, the most basic application of the probabilistic method constructs a random variable X to measure a single parameter, and calculates its expectation for a random construction. Then, as we did above for maximum cut, one can conclude that there is an outcome in which X takes a value which is at least its expectation.

Part of the appeal of judicious partitioning problems thus stems from the fact that they push the envelope for probabilistic methods: when one needs to satisfy two separate parameters X and Y simultaneously, the above argument breaks down, because $X \ge \mathbb{E}[X]$ and $Y \ge \mathbb{E}[Y]$ do not necessarily hold at the same time. It is therefore no coincidence that many arguments in the area of judicious partitioning blend techniques from probabilistic and extremal combinatorics, using variance calculations or martingale concentration inequalities to strengthen conclusions drawn from the initial first-moment calculations. This area has attracted many researchers, who have now produced a wealth of results, covering combinatorial structures spanning graphs, directed graphs, and hypergraphs. We direct the interested reader to any of [3, 5, 7, 8, 16, 19, 22, 23] (by no means a comprehensive list), or to either of the surveys [6, 25].

1.1 Large bisections

In this paper, we focus on bisections. Compared to cuts, bisections are much more complicated to analyze. For example, no bisection analogue to Theorem 1.1 was previously known. Also, in Computer Science, even long after Goemans and Williamson [14] discovered a 0.878-factor approximation to Max Cut via semi-definite programming, the best Max Bisection results still lag behind, from the 0.651-factor approximation of Frieze and Jerrum [13] that it inspired, to the current best 0.703-factor approximation of Feige and Langberg [12].

Note that when m is linear in n, the Edwards bound (1) for cuts is relatively weak, giving only

$$\text{Max Cut} \geq \frac{m}{2} + \Omega(\sqrt{n}) \,.$$

As cuts have been more successfully studied than bisections, there is an existing result which still has useful implications for sparse graphs. As noticed in [24, 11], the results that Edwards proved in [10] implicitly imply that connected graphs admit a bipartition of size at least

$$\text{Max Cut} \geq \frac{m}{2} + \frac{n-1}{4} \,. \tag{2}$$

As a similar result, Erdős, Gyárfás, and Kohayakawa [11] established the following bound for graphs without isolated vertices:

$$\text{Max Cut} \geq \frac{m}{2} + \frac{n}{6} \,, \tag{3}$$

which can be proved by using the probabilistic method to produce a suitable ordering of the vertices to feed into the greedy algorithm alluded to in the beginning of the Introduction.

Unfortunately, neither result transfers directly, because every bisection of the star $K_{1,n-1}$ has size $\frac{n}{2} = \frac{m+1}{2}$. This situation can be circumvented by imposing some minor conditions. To state our result, we introduce the following new notion, with respect to which we can prove sharp bounds on Max Bisection.

Definition 1.2. *Let T be a connected component of a given graph. We say that T is a **tight component** if it has the following properties.*

(i) *For every vertex $v \in T$, the subgraph induced by $T \setminus \{v\}$ contains a perfect matching.*

(ii) *For every vertex $v \in T$ and every perfect matching of $T \setminus \{v\}$, no edge of the matching has exactly one endpoint adjacent to v.*

We can now define the parameter τ to be the number of tight components in a graph. Our first result is a very precise extension of (2) and (3) to bisections.

Theorem 1.3. *Let G be a graph of maximum degree Δ which has τ tight components. Then there is a bisection of size at least*

$$\frac{m}{2} + \frac{n - \max\{\tau, \Delta - 1\}}{4}.$$

This theorem is tight in both parameters τ and Δ. For example if the graph G consists of $n/3$ vertex-disjoint copies of a triangle, then $m = n, \tau = n/3, \Delta = 2$, and one can easily see that the largest bisection is of size $2n/3 = m/2 + n/6 = m/2 + (n - \tau)/4$. As another example, if the graph is $K_{1,n-1}$ for an even integer n, then $m = n - 1, \tau = 0, \Delta = n - 1$, and the largest bisection is of size $n/2 = m/2 + 1/2 = m/2 + (n - \Delta + 1)/4$. Many other tight examples can be constructed.

Analogues to (2) and (3) follow as easy corollaries from Theorem 1.3.

Corollary 1.4. *For $n \geq 2$, every n-vertex connected graph with m edges and maximum degree Δ has a bisection of size at least $\frac{m}{2} + \frac{n+1-\Delta}{4}$.*

Corollary 1.5. *Every n-vertex graph with m edges, maximum degree at most $\frac{n}{3} + 1$, and no isolated vertices, has a bisection of size at least $\frac{m}{2} + \frac{n}{6}$.*

More importantly, the ideas used to prove Theorem 1.3 appear to be fairly robust. Variants on the theme produce several additional results, one of which provides a parameterized interpolation between cut and bisection results. Let an *α-bisection* be a cut in which both sides have at least $(\frac{1}{2} - \alpha)n$ vertices.

Theorem 1.6. *Let $0 \leq \alpha \leq \frac{1}{6}$ be a fixed parameter. Then every n-vertex graph with m edges and no isolated vertices contains an α-bisection of size at least $\frac{m}{2} + \alpha n$.*

Remark. This provides further insight to the bound $\frac{m}{2} + \frac{n}{6}$ of (3), since it shows that the bound can in fact always be achieved by a $\frac{1}{6}$-bisection.

1.2 Judicious bisections

A *judicious bisection* of a graph is a bisection in which both parts span few edges. This is a qualitative definition, and a quantitative definition will be given for each problem that we consider. There is a nontrivial obstacle to extending results from judicious cuts to judicious bisections, which is the fact that the direct analogue is false. Indeed, if one considers the star $K_{1,n-1}$, one observes that although the natural bipartition produces an excellent judicious cut which spans no edges on either side, every *bisection* spans exactly $\frac{n}{2} - 1$ edges on one side, and none on the other. This is far from the ideal of $\big(\frac{1}{4} + o(1)\big)m$ edges which is suggested by an expected value calculation for a random bisection.

To circumvent this issue, Bollobás and Scott proposed several conditions under which they suspected the ideal result would hold. Specifically, in their survey [6], they suggested that perhaps one of $\delta \to \infty$ or $\Delta = o(n)$ might be enough. (Here and in the remainder, δ and Δ correspond to the minimum and maximum degree of the graph.) It is clear that these conditions exclude the star $K_{1,n-1}$, together with similar graphs which also are obstructions. Our first result proves their conjecture.

Theorem 1.7. *Let ε be a fixed positive constant and let G be an n-vertex graph with m edges such that **(i)** $m \geq \varepsilon^{-2}n$, or **(ii)** all degrees are at most $\frac{\varepsilon^2 n}{2}$. If n is sufficiently large, then G admits a bisection $V = V_1 \cup V_2$ in which each V_i spans at most $\big(\frac{1}{4} + \varepsilon\big)m$ edges.*

Continuing the study of judicious bisection of graphs with large minimum degree, Bollobás and Scott suggested that once the minimum degree was at least 2, one could substantially improve upon the behavior of the now-excluded pathological $K_{1,n-1}$ example. Specifically, they conjectured in [6] that such graphs always have bisections spanning at most $\frac{m}{3}$ edges in each part. This result would be tight, for example in the case when the graph was a triangle. (See the discussion after Theorem 1.9 for further examples.) Our next theorem asymptotically confirms Bollobás and Scott's conjecture. Here and in the remainder, $e(U)$ denotes the number of edges spanned by the set U.

Theorem 1.8. *Every n-vertex graph $G = (V, E)$ with m edges and minimum degree at least 2 has a bisection in which both*

$$e(V_i) \leq \left(\frac{1}{3} + o(1)\right) m .$$

In [6, 25], they further asked how the bound changes as the minimum degree condition grows, but did not conjecture the value of the constant. By Theorem 1.7, the asymptotic fraction should shrink to $\frac{1}{4}$ with larger δ. Our next result determines the asymptotic constants precisely, providing a complete answer to the question of Bollobás and Scott.

Theorem 1.9. *Let δ be a positive even integer. Every n-vertex graph $G = (V, E)$ with m edges and minimum degree at least δ has a bisection in which both*

$$e(V_i) \le \left(\frac{\delta + 2}{4(\delta + 1)} + o(1) \right) m \,.$$

Remark. For odd δ, one can deduce a bound by applying the above result for the even integer $\delta - 1$, and this is asymptotically tight (see below). The dependence between the constant and δ is therefore rather non-trivial.

Adding to the challenge, it turns out that the extremal examples come from at least two strikingly different families. For the first family, take the vertex-disjoint union of an arbitrary number of cliques of order δ, together with an arbitrary number of cliques of order $\delta+1$, and add a new vertex adjacent to all vertices in the cliques. For the second family, consider the complete bipartite graph $K_{\delta+1,n-\delta-1}$. Note that we obtain graphs with minimum degree $\delta + 1$ (odd) from the second family of constructions, as well as from the first family of constructions when only $K_{\delta+1}$ are used. These then provide examples for asymptotic tightness in the remark above.

Proposition 1.10. *Constructions from both of the above families only admit bisections which span at least $\left(\frac{\delta+2}{4(\delta+1)} - o(1)\right)m$ edges on some side.*

The main tool in proving Theorems 1.8 and 1.9 is the following strengthening of Theorem 1.7 for sparse graphs, which can also be viewed as a "randomized" version of Theorem 1.3. It builds upon the proof of Theorem 1.3 and uses martingale concentration techniques to produce a bisection which is not only large, but also judicious. The key benefit is its parameterization in terms of the number of tight components.

Theorem 1.11. *Given any real constants $C, \varepsilon > 0$, there exist $\gamma, n_0 > 0$ for which the following holds. Every graph $G = (V, E)$ with $n \ge n_0$ vertices, $m \le Cn$ edges, maximum degree at most γn, and τ tight components, admits a bisection $V = V_1 \cup V_2$ where each V_i induces at most*

$$\frac{m}{4} - \frac{n - \tau}{8} + \varepsilon n$$

edges.

To make use of this theorem for graphs whose maximum degree is not bounded, we first identify the large-degree vertices, say, of degree at least $n^{3/4}$, and find an optimal split of these vertices. Then armed with the appropriate parameterization, the remainder of the proof follows from a (somewhat intricate) analysis of the relationship between tight components, the number of edges, large-degree vertices, and the minimum degree condition.

1.3 Related bisection results

In the process of extending classical results from cuts to bisections, we also discovered several related statements which we present in this section. The judicious bipartition bound in Theorem 1.1 is tight for complete graphs, and therefore cannot be improved in general. This is clear: if the maximum cut only has size $(\frac{1}{2} + o(1))m$, then no bipartition can induce substantially fewer than $\frac{m}{4}$ edges on each side. There is only potential for improvement if the graph already has a cut which is larger than the guaranteed $\frac{m}{2}$. In this setting, Alon, Bollobás, Krivelevich, and Sudakov [1] proved that when the maximum cut is large, then indeed there is a judicious bipartition in which each side induces substantially fewer than $\frac{m}{4}$ edges.

Bollobás and Scott [8] asked whether the same is also true for judicious bisections; we show that this is not the case.

Proposition 1.12. *For arbitrarily large m, there are m-edge graphs with bisections of size at least $\frac{3}{4}m$, but with $\max\{e(V_1), e(V_2)\} \geq \frac{m}{4}$ for every bisection $V_1 \cup V_2$.*

Several researchers [2, 20, 21] also noticed that if a graph has chromatic number at most k, then it admits a bipartition of size at least $\frac{k+1}{2k}m$ when k is odd and $\frac{k}{2(k-1)}m$ when k is even. This unfortunately is false for bisections, as the star $K_{1,n-1}$ has chromatic number 2, but maximum bisection $\frac{n}{2}$, not $\frac{2n}{3}$. Instead, we prove an almost-bisection result under a maximum degree condition.

Theorem 1.13. *For any positive integer r, if G is a graph with maximum degree at most r, then it has a bipartition of size at least $\frac{r+1}{2r}m$ when r is odd and $\frac{r+2}{2(r+1)}m$ when r is even, in which the part sizes differ by at most $\frac{r}{2} + 1$.*

An obvious corollary of Theorem 1.13 is that if G is a graph with maximum degree at most r, then it has a true bisection of size at least $\frac{r+1}{2r}m - \frac{r(r+1)}{4}$ when r is odd and $\frac{r+2}{2(r+1)}m - \frac{r(r+2)}{4}$ when r is even. We also give a simple proof of the following result of Bollobás and Scott [7].

Theorem 1.14. *For any positive integer r, every r-regular graph has a bisection of size at least $\frac{r+1}{2r}m$ when r is odd and $\frac{r+2}{2(r+1)}m$ when r is even.*

1.4 Notation and organization

Since the majority of our results are asymptotic in nature, we will implicitly ignore rounding effects (e.g., when bisecting graphs of odd order) whenever these effects are of smaller order than our error terms. The following (standard) asymptotic notation will be utilized extensively. For two functions $f(n)$ and $g(n)$, we write $f(n) = o(g(n))$, $g(n) = \omega(f(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ if there exists a constant M such that $|f(n)| \leq M|g(n)|$ for all sufficiently large n. We also write $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ are satisfied.

This paper is organized as follows. In the next section, we show how Theorem 1.7 follows from a rather short second-moment argument. In Section 3, we study tight components, and develop the main tools needed to prove Theorems 1.3 and 1.6, and Corollaries 1.5 and 1.4. In Section 4, we further combine these tools with martingale concentration results and prove Theorem 1.11. We move to our main result in Sections 5 and 6, where we first prove the $\delta = 2$ case (Theorem 1.8), and then the general case (Theorem 1.9). We further confirm its asymptotic optimality by proving Proposition 1.10 in Section 7. Section 8 contains the proofs of the various related results introduced in Section 1.3. Finally, we conclude with some remarks.

2 The basic second-moment argument

We begin by answering two questions of Bollobás and Scott, regarding settings in which it was suspected that a judicious bisection would exist. These were the settings where (i) the minimum degree grew with n, or (ii) the maximum degree was of order $o(n)$. Either condition would rule out the basic pathological counterexample, which was based on a star.

It turns out that a second-moment argument suffices to answer these questions. The most elementary bisection algorithm constructs an arbitrary pairing of the vertices, and then separates each pair across $V_1 \cup V_2$, independently and uniformly at random. We will now show that this performs well in both of the above settings.

Lemma 2.1. *Let $G = (V, E)$ be an n-vertex graph with m edges, and define*

$$\Lambda = \frac{1}{16}\left(3m + \sum_{v \in V} d(v)^2\right), \tag{4}$$

where $d(v)$ denotes the degree of vertex v. Then G admits a bisection $V = V_1 \cup V_2$ for which each

$$e(V_i) \le \frac{m}{4} + \sqrt{2\Lambda}\,.$$

Proof. Define the random variables Y_1 and Y_2 to be the numbers of edges induced by each V_i after the elementary random bisection algorithm, described above. It is clear that $\mathbb{E}[Y_1] \le \frac{m}{4}$, because for each edge $e = \{v, w\}$, $\mathbb{P}(v, w \in V_1) = 0$ if v, w were initially paired, and it is $\frac{1}{4}$ otherwise.

To calculate the variance of Y_1, define an indicator variable I_e for each edge e, which takes the value 1 precisely when both endpoints of e fall in V_1. Note that for any e whose endpoints were initially paired, I_e is always 0. Thus, such e will never contribute to Y_1; let F be the set of all other edges (i.e., edges whose endpoints were not initially paired). We then have

$$\begin{aligned}
\mathbb{E}\left[(Y_1 - \mathbb{E}[Y_1])^2\right] &= \mathbb{E}\left[\left(\sum_{e \in F}(I_e - \mathbb{E}[I_e])\right)^2\right] \\
&= \mathbb{E}\left[\sum_{e \in F}(I_e - \mathbb{E}[I_e])^2 + \sum_{e \ne f \in F}(I_e - \mathbb{E}[I_e])(I_f - \mathbb{E}[I_f])\right] \\
&= \sum_{e \in F}\left(\mathbb{E}\left[I_e^2\right] - \mathbb{E}[I_e]^2\right) + \sum_{e \ne f \in F}\left(\mathbb{E}[I_e I_f] - \mathbb{E}[I_e]\,\mathbb{E}[I_f]\right)
\end{aligned}$$

Each term in the first sum is precisely $\frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{3}{16}$. To estimate the second sum, observe that I_e and I_f are independent whenever the two pairs which hold the endpoints of e are disjoint from the two pairs which hold the endpoints of f. In those cases, $\mathbb{E}[I_e I_f] - \mathbb{E}[I_e]\,\mathbb{E}[I_f] = 0$. Otherwise, the endpoints of e and f are contained in either 2 or 3 pairs. If only two pairs are involved, it is easy to see that it is impossible for both of $e \ne f$ to lie in V_1, so $\mathbb{E}[I_e I_f] - \mathbb{E}[I_e]\,\mathbb{E}[I_f] = -\left(\frac{1}{4}\right)^2$. If three pairs are involved but e and f do not share a vertex, it is also impossible for both e and f to lie in V_1, so again $\mathbb{E}[I_e I_f] - \mathbb{E}[I_e]\,\mathbb{E}[I_f] = -\left(\frac{1}{4}\right)^2$. The final possibility has e and f sharing a common endpoint, and their 3 total vertices lie in 3 different pairs. Then, the probability of e and f both falling in V_1 is $\frac{1}{8}$, and hence $\mathbb{E}[I_e I_f] - \mathbb{E}[I_e]\,\mathbb{E}[I_f] \le \frac{1}{8} - \left(\frac{1}{4}\right)^2 = \frac{1}{16}$. Therefore,

$$\mathrm{Var}[Y_1] \le \sum_{e \in F} \frac{3}{16} + \sum_{e \ne f,\ \text{incident}} \frac{1}{16} < \frac{3}{16}|F| + \frac{1}{16} \sum_{v \in V} d(v)^2 \le \Lambda\,,$$

where the strict inequality comes from writing $d(v)^2$ instead of $d(v)(d(v)-1)$. Chebyshev's inequality then gives

$$\mathbb{P}\left(Y_1 \ge \mathbb{E}[Y_1] + \sqrt{2\Lambda}\right) \le \frac{\mathrm{Var}[Y_1]}{2\Lambda} < \frac{1}{2}\,.$$

The same inequality holds for Y_2 by symmetry, so we conclude that there is an outcome in which both $e(V_i) \le \frac{m}{4} + \sqrt{2\Lambda}$. □

We now proceed to prove Theorem 1.7, which considers the situations when (i) the number of edges is $\omega(n)$, or (ii) the maximum degree is $o(n)$.

Proof of Theorem 1.7. Let $\varepsilon > 0$ be given. We begin with (i). We will show that whenever there are $m \ge \varepsilon^{-2} n$ edges and n is sufficiently large, there is a judicious bisection in which each $e(V_i) \le \left(\frac{1}{4} + \varepsilon\right)m$. Indeed, then the trivial bound $d(v) < n$ applied to (4) gives

$$\Lambda < \frac{1}{16}\left(3m + n \sum_{v \in V} d(v)\right) = \frac{1}{16}\left(3m + n(2m)\right)\,,$$

which is well below $\frac{1}{2}mn$ for large n. Using $m \ge \varepsilon^{-2} n$, we find that $\Lambda < \frac{1}{2}\varepsilon^2 m^2$, which together with Lemma 2.1 implies that there is a bisection in which each $e(V_i) \le \frac{m}{4} + \sqrt{2 \cdot \frac{1}{2}\varepsilon^2 m^2}$, as desired.

For (ii), we instead assume that $d(v) \le \frac{\varepsilon^2 n}{2}$. Substituting our new bound for $d(v)$ into (4), we find that

$$\Lambda < \frac{1}{16}\left(3m + \left(\frac{\varepsilon^2 n}{2}\right) \sum_{v \in V} d(v)\right) = \frac{1}{16}\left(3m + \varepsilon^2 nm\right)\,,$$

which is well below $\frac{1}{8}\varepsilon^2 nm$ for large n. If $m \ge \frac{n}{4}$, then we have $\Lambda < \frac{1}{2}\varepsilon^2 m^2$, which completes the proof as for (i) above. On the other hand, if $m < \frac{n}{4}$, then there are at least $\frac{n}{2}$ isolated vertices. We may then apply the original result of Bollobás and Scott (Theorem 1.1 in the Introduction) to the graph induced by the non-isolated vertices, obtaining a cut (not necessarily a bisection) with each $e(V_i) \le \left(\frac{1}{4} + \varepsilon\right)m$. This is easily converted into a perfect bisection by suitably distributing the $\ge \frac{n}{2}$ isolated vertices. □

3 Large bisection

Our eventual goal is to prove Theorem 1.9, which completely answers the question of judicious bisections under precise minimum-degree conditions. Using Theorem 1.7 as a starting point, we see that the difficult case is when both the total number of edges and the maximum degree are linear in n. The natural route is therefore to pre-allocate the vertices of large degree, and apply randomized arguments on the remainder of the graph. In this section, we develop tools which we apply to the graph after the removal of large degree vertices. Our first objective is to establish a gain above the trivial bound for the size of a bisection. (This is the precursor to a judicious bisection.) Specifically, we prove Theorems 1.3 and 1.6.

The basic idea is to execute a deterministic greedy algorithm on a suitably ordered input. Indeed, suppose the vertices have been partitioned into pairs $P_1, \ldots, P_{n/2}$. We then obtain a bisection $V = V_1 \cup V_2$ by separating each pair. Start by arbitrarily splitting P_1, and iterate as follows. When considering $P_i = \{v, w\}$, there are two ways to split: either we place $v \in V_1$ and $w \in V_2$, or we place $v \in V_2$ and $w \in V_1$. Let x_{i-1} be the number of crossing edges in the bisection induced by the previously generated partition of $P_1 \cup \ldots \cup P_{i-1}$. If x_i is the number after splitting P_i, then an averaging argument shows that there is a way to split P_i such that

$$x_i - x_{i-1} \geq \frac{1}{2} \left[e(P_1 \cup \cdots \cup P_i) - e(P_1 \cup \cdots \cup P_{i-1}) \right] \tag{5}$$

Our algorithm proceeds by always selecting the split which maximizes $x_i - x_{i-1}$. Observe that this immediately implies that there is a bisection of size at least $\frac{m}{2}$. Our objective is now to improve upon this bound by taking advantage of certain situations in which (5) is not tight.

For example, if $P_i = \{v, w\}$ was in fact an edge, then both ways of splitting P_i would capture that edge. Inequality (5) only assumes that at least one of the splits captures the edge, so in this case we could actually improve the bound to

$$x_i - x_{i-1} \geq \frac{1}{2} \left[e(P_1 \cup \cdots \cup P_i) - e(P_1 \cup \cdots \cup P_{i-1}) \right] + \frac{1}{2} \tag{6}$$

Alternatively, even if $P_i = \{v, w\}$ were not an edge, there is another way to obtain the same bound as in (6). Indeed, whenever $e(P_1 \cup \cdots \cup P_i) - e(P_1 \cup \cdots \cup P_{i-1})$ is an odd number, it is impossible for (5) to be sharp, because the right hand side is not an integer. In such a situation, we automatically obtain (6). Note that this happens when the "back-degrees", i.e., degrees of v and w in $P_1 \cup \cdots \cup P_{i-1}$ are of different parity.

These two situations are advantageous because if (6) holds a linear number of times, then we obtain our desired gain over the trivial bound by an amount linear in n.

3.1 Tight components and free vertices

We aim to exploit the two possibilities outlined at the end of the previous section. In order to take full advantage of the first, we begin by taking a matching of maximum size as the basis of our pairing. In this section, we demonstrate the connection between maximum matchings and tight components.

Recall from the introduction (Definition 1.2) that a tight component is a connected component T such that for every $v \in T$, the subgraph induced by $T \setminus \{v\}$ contains a perfect matching, and every perfect matching of $T \setminus \{v\}$ has the property that no edge of the perfect matching has exactly one endpoint adjacent to v.

Definition 3.1. *Let $\{e_1, \ldots, e_s\}$ be the edges of a maximum matching in a graph $G = (V, E)$, and let W be the set of vertices not in the matching. With respect to this fixed matching, say that a vertex v in a matching edge e_i is a* ***free neighbor*** *of a vertex $w \in W$ if w is adjacent to v, but w is not adjacent to the other endpoint of e_i. Call a vertex $w \in W$ a* ***free vertex*** *if it has at least one free neighbor.*

The following lemma delineates the relationship between free vertices and tight components.

Lemma 3.2. *Let $\{e_1, \ldots, e_s\}$ be the edges of a maximum matching in a graph $G = (V, E)$, and let W be the set of vertices not in the matching. Further assume that among all matchings of maximum size, we have chosen one which maximizes the number of free vertices in W. Then, every tight component contains a distinct non-free vertex of W, and all non-free W-vertices are covered in this way (there is a bijective correspondence).*

Remark. In particular, the number of non-free vertices of W is equal to the number of tight components of G.

Proof. The matching $\{e_1, \ldots, e_s\}$ must be maximal within each connected component. One basic property of a tight component is that it contains an almost-perfect matching which misses only one vertex. Consequently, by maximality, in every tight component T, $\{e_1, \ldots, e_s\}$ must miss exactly one vertex $w \in W$. Furthermore, the second property of a tight component is that w must have either 0 or 2 neighbors in each edge e_i in T (and w must have 0 neighbors in each edge e_j not in T, since T is the connected component containing w). Therefore, the unique vertex w is in fact a non-free W-vertex contained in T.

The remainder of the proof concentrates on the more substantial part of the claim, which is that each non-free W-vertex is contained in some tight component. Consider such a vertex w, and let T be a maximal set of vertices which (i) contains w, (ii) induces a connected graph which is a tight component, and (iii) does not cut any e_i. Since the set $\{w\}$ satisfies (i)–(iii), our optimum is taken over a non-empty set, and so T exists.

If T is already disconnected from the rest of the graph, then we are done. So, consider a vertex $v \notin T$ which has a neighbor $v' \in T$. If $v \in W$, then we can modify our matching by taking the edge vv', and changing the matching within T by using property (ii) to generate a new matching of $T \setminus \{v'\}$. This will not affect the matching outside of $T \cup \{v\}$, because property (iii) insulates the adjustments within T from the rest of the matching outside. We would then obtain a matching with one more edge, contradicting maximality. Therefore, all vertices $v \notin T$ which have neighbors in T also satisfy $v \notin W$.

Let us then consider a vertex $v_1 \notin T$ with a neighbor $v' \in T$. We now know that v_1 must be covered by a matching edge; let v_2 be the other endpoint of that edge. By (iii), we also have $v_2 \notin T$. Note that v_2 cannot have a neighbor $w' \in W \setminus T$, or else we could improve our matching by replacing v_1v_2 with $w'v_2$ and v_1v', and then using (ii) to take a perfect matching of $T \setminus \{v'\}$.

Our next claim is that v_2 must be adjacent to v' as well. Indeed, assume for contradiction that this is not the case. Then, consider modifying our matching by replacing v_1v_2 with the edge v_1v' and changing the matching within T by using (ii) to generate a new matching of $T \setminus \{v'\}$. As before, (iii) ensures that the result is still a matching. This time, the new matching has the same size as the original one, but with more free W-vertices (note that the vertex v_2 replaced the vertex w in the set W). To see this, observe that v_2 is now unmatched and free because it is adjacent to v_1 but not v'. Previously, the only W-vertex inside T was our original w, which we assumed to be non-free in the first place. Also, no other vertices outside of T changed from being free to non-free, because we already showed that no W-vertices outside of T were adjacent to $T \cup \{v_2\}$, and so any vertex that was free by virtue of its adjacency with v_1 but not v_2 is still free because it is not adjacent to v' either. This contradiction to maximality establishes that v_2 must be adjacent to v'.

We now have v_1, v_2, and v' all adjacent to each other, and no vertices of $W \setminus T$ are adjacent to $T \cup \{v_1, v_2\}$. Our argument also shows that for any $v'' \in T$ which is adjacent to one of v_1 or v_2, it also must be adjacent to the other. Our final objective is to show that $T' = T \cup \{v_1, v_2\}$ also satisfies (i)–(iii), which would contradict the maximality of T. Properties (i) and (iii) are immediate, so it remains to verify the conditions of a tight component. Since T is tight and v_1v_2 is an edge, $T' \setminus \{u\}$ has a perfect matching for any $u \in T$. The tightness of T and the pairwise adjacency of v_1, v_2, and v' also produce this conclusion if $u \in \{v_1, v_2\}$. It remains to show that for any $u \in T'$ and any perfect matching of $T' \setminus \{u\}$, u has either 0 or 2 vertices in every matching edge. But if this were not the case, then we could replace the matching within T' with the violating matching of $T' \setminus \{u\}$. The two matchings would have the same size, but u would become a free vertex. No other vertex of W is adjacent to T' by our observation above, so the number of free vertices would increase, contradicting the maximality of our initial matching. Therefore, T' induces a tight component, contradicting the maximality of T. We conclude that T must have been disconnected from the rest of the graph, as required. □

3.2 Proof of Theorem 1.3

In this section, we show that every n-vertex graph with m edges, maximum degree Δ, and exactly τ tight components has a bisection of size at least

$$\frac{m}{2} + \frac{n - \max\{\tau, \Delta - 1\}}{4}.$$

Our proof is somewhat similar to that used by Erdős, Gyárfas, and Kohayakawa [11] to estimate Max Cut.

Start by taking a maximum matching $\{e_1, \ldots, e_s\}$ as in Lemma 3.2, which further maximizes the number of free vertices in $W = V \setminus \{e_1, \ldots, e_s\} = \{w_1, \ldots, w_r\}$. Note that $n = r + 2s$. By maximality, W is an independent set, and there cannot be any $e_i = \{v, v'\}$ such that vw and $v'w'$ are both edges, with w, w' distinct vertices in W. The second observation shows that if two vertices $w, w' \in W$ each have a free neighbor in an e_i, then their free neighbor is the same vertex.

Let us now partition the vertices of V into disjoint pairs. Later, we will distribute the vertices by separating each pair; this will produce a bisection. First, form a pair from each matching edge by taking its endpoints. In a slight abuse of notation, we will denote these pairs by $e_1, \ldots, e_s$. Then, for the vertices in W, greedily form disjoint pairs by taking vertices w_i and w_j whose sets of free neighbors are distinct. (Note that if w_i has no free neighbors and w_j has several, we may pair them.) Let $p_1, \ldots, p_{r'}$ be the pairs constructed from W in this way. Note that at the end, we must have $r' \le r/2$, and all of the $n - 2(s + r')$ remaining vertices must have exactly the same set S of free neighbors. Arbitrarily collect the remaining unpaired vertices into pairs $q_1, \ldots, q_t$, where $t = \frac{n}{2} - (s + r')$.

If $S = \emptyset$, then all remaining vertices are non-free, so Lemma 3.2 implies that $n - 2(s + r') \le \tau$, where τ is the number of tight components. Otherwise, there is some vertex $v \in S$ which is adjacent to all remaining vertices. Since the degree of v is at most Δ and one of its incident edges is the matching edge which covers it, we find that $n - 2(s + r') \le \Delta - 1$. Therefore, $n - 2(s + r') \le \max\{\tau, \Delta - 1\}$, and so it suffices to show that there exists a bisection of size at least $\frac{m}{2} + \frac{s+r'}{2}$ because

$$\frac{m}{2} + \frac{s + r'}{2} \ge \frac{m}{2} + \frac{n - \max\{\tau, \Delta - 1\}}{4}.$$

For this, we order the pairs. Start by taking $e_1, e_2, \ldots, e_s$ in that order. We iteratively insert the pairs p_i into the sequence as follows. By construction, for each p_i, there exists an edge e_j such that only one of the vertices in p_i has a free neighbor in e_j. Let e_j be the lowest-indexed such edge and place p_i between e_j and e_{j+1} (arbitrarily order the pairs p_i which are placed between the same consecutive edges). Finally, append the pairs $q_1, \ldots, q_t$ at the end of the sequence, and call the resulting sequence $P_1, P_2, \ldots, P_{n/2}$. (Here, each P_i is either an e_j, a p_j, or a q_j.)

We split each pair across the bisection by using the P_i-sequence as the input to the greedy algorithm presented at the beginning of Section 3. By our discussion there, it suffices to show that (6) holds at least $s + r'$ times. We already have s pairs P_i of the form e_j (edges), which we know from the beginning to produce (6).

We also have r' pairs P_i of the form $p_j = \{w, w'\}$. There are no edges from w or w' back to any vertices of W in $P_1 \cup \cdots \cup P_{i-1}$, because W is an independent set. As for edges e_j which appear among $P_1 \cup \cdots \cup P_{i-1}$, whenever w (or w') has exactly 1 edge to an endpoint of e_j, then that endpoint is a free neighbor of w (respectively. w'). By construction, w and w' have exactly the same set of free neighbors in $P_1 \cup \cdots \cup P_{i-1}$, except for one endpoint of an edge, at which they differ. Hence the total number of edges from w and w' back to $P_1 \cup \cdots \cup P_{i-1}$ is odd, which we also showed at the beginning to give (6).

Therefore, we indeed achieve (6) at least $s + r'$ times, and so the final bisection has size at least $\frac{m}{2} + \frac{s+r'}{2}$, as required. □

3.3 Bisection corollaries

We now use Theorem 1.3 to prove Corollaries 1.5 and 1.4. We begin by showing that every n-vertex graph with m edges, maximum degree at most $\frac{n}{3} + 1$, and no isolated vertices, has a bisection of size at least $\frac{m}{2} + \frac{n}{6}$.

Proof of Corollary 1.5. Tight components are always of odd order, because the deletion of any vertex leaves a graph which has a perfect matching. Since we assume there are no isolated vertices, this forces the number of tight components to be at most $n/3$, since the smallest possible tight component is then of order 3. Applying Theorem 1.3 with $\Delta \leq \frac{n}{3} + 1$ and $\tau \leq \frac{n}{3}$, we find that G has a bisection of size at least

$$\frac{m}{2} + \frac{n - (n/3)}{4} = \frac{m}{2} + \frac{n}{6}\,,$$

as desired. □

Next, we show that every n-vertex connected graph with m edges and maximum degree Δ has a bisection of size at least $\frac{m}{2} + \frac{n+1-\Delta}{4}$.

Proof of Corollary 1.4. If $n = 2$, then the graph consists of a single edge, and the conclusion can easily be checked to be true. Thus we may assume $n \geq 3$, from which we obtain $\Delta \geq 2$, since the graph is connected. Moreover, since the graph is connected, the number of tight components is either 0 or 1. So $\max\{\tau, \Delta - 1\} = \Delta - 1$. Therefore, Theorem 1.3 produces a bisection of size at least

$$\frac{m}{2} + \frac{n - (\Delta - 1)}{4}\,,$$

as desired. □

3.4 Approximate bisection

Our general approach from Section 3.2 turns out to be quite powerful, as a slight twist of the approach produces Theorem 1.6, which concerns α-bisections (bipartitions with both sides containing at least $(\frac{1}{2} - \alpha)n$ vertices). In this section, we prove that for any $0 \le \alpha \le \frac{1}{6}$, every graph which has no isolated vertices contains an α-bisection of size at least $\frac{m}{2} + \alpha n$.

Proof of Theorem 1.6. As in Section 3.2, take an optimal matching $\{e_1, \ldots, e_s\}$, greedily form the pairs $p_1, \ldots, p_{r'}$, and insert them into the sequence of e_i's as before. However, this time we will not necessarily pair up the remaining vertices.

Case 1: There are at most $\frac{n}{3}$ unpaired vertices. The number of unpaired vertices is exactly $n - 2(s + r')$, so

$$\begin{aligned} n - 2(s + r') &\le \frac{n}{3} \\ s + r' &\ge \frac{n}{3}\,. \end{aligned}$$

In this case, we now arbitrarily pair up the remaining vertices and append the new pairs to the end of the sequence of pairs. Applying the same procedure and analysis as before, we obtain a *perfect* bisection of size at least

$$\frac{m}{2} + \frac{s + r'}{2} \ge \frac{m}{2} + \frac{n}{6} \ge \frac{m}{2} + \alpha n\,,$$

as desired.

Case 2: There are more than $\frac{n}{3}$ unpaired vertices. Recall that the remaining unpaired vertices are either all non-free, or all have the same nonempty set of free neighbors. Lemma 3.2, bounds the number of non-free vertices by the number of tight components. Since all tight components are odd, and in this case we have no isolated vertices, the number of tight components is at most $\frac{n}{3}$. Therefore, we must have that all unpaired vertices have the same nonempty set of free neighbors.

Thus far, we have interleaved the p_i into the original sequence of e_i's, and some vertices, say $w_1, \ldots, w_k$ with $k = n - 2(s + r') > \frac{n}{3}$, remain. Importantly, note that both of $\frac{k}{2} \pm \alpha n$ are between 0 and k inclusive.

Let e_i be the first matching edge which contains a free neighbor of the remaining vertices. Apply the greedy splitting algorithm (beginning of Section 3) to the sequence of pairs up through e_i. Let v be the endpoint of e_i that is adjacent to all of $R = \{w_1, \ldots, w_k\}$. Place $\frac{k}{2} + \alpha n$ vertices of R on the side opposite v, and the other $\frac{k}{2} - \alpha n$ vertices of R on the same side as v. Then apply the greedy splitting algorithm to the rest of the pairs in the sequence.

Note that by the trivial bound (5), every split always adds at least half of the new edges to the cut, and the cut we obtain in the end has parts of size $(\frac{1}{2} \pm \alpha)n$. To finish the proof, we will show that in the step when we split the vertices of R, there is an extra gain of αn edges over the trivial bound of (5). For this, observe that vertices of R are adjacent to neither or both of the endpoints of each e_j, $j < i$, but adjacent to $v \in e_i$. By maximality of the original matching, W is an independent set, so there will be no edges from R to the pairs p_j preceding e_i. Let b be the number of edges e_j, $j < i$, which have both endpoints adjacent to some vertex of R.

Regardless of how we split the vertices of R, exactly b edges out of the $2b$ edges that are incident to e_j (for $j < i$) will cross the cut. If a vertex $w \in R$ is placed on the same side as $v \in e_i$, then it will add no edge to the cut, and if it is placed on the opposite side of v, then it will add one edge to the cut. Therefore, by placing $\frac{k}{2} + \alpha n$ vertices of R on the opposite side of v, the total number of crossing edges from R back to the pairs preceding it is exactly

$$b + \left(\frac{k}{2} + \alpha n\right) = \left(b + \frac{k}{2}\right) + \alpha n$$

Since there were exactly $2b + k$ edges from R back to the preceding pairs, this produces the desired gain of αn. □

4 Judicious bisection

Our previous results produced bipartitions which achieved certain cut sizes. In this section, we use the following martingale concentration result (essentially the Hoeffding-Azuma inequality) to prove the stronger Theorem 1.11, which produces bisections that induce few edges in each part.

Theorem 4.1. *(Corollary 2.27 in [18].) Given real numbers $\lambda, C_1, \ldots, C_n > 0$, let $f : \{0,1\}^n \to \mathbb{R}$ be a function satisfying the following Lipschitz condition: whenever two vectors $z, z' \in \{0,1\}^n$ differ only in the i-th coordinate (for any i), we always have $|f(z) - f(z')| \le C_i$. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables, each taking values in $\{0,1\}$. Then, the random variable $Y = f(X_1, \ldots, X_n)$ satisfies*

$$\mathbb{P}\left(|Y - \mathbb{E}[Y]| \ge \lambda\right) \le 2 \exp\left\{-\frac{\lambda^2}{2\sum C_i^2}\right\}.$$

4.1 Core result

We use this to control the performance of a randomized partitioning algorithm, which will produce Theorem 1.11 (restated here for the reader's convenience):

Theorem. *Given any real constants $C, \varepsilon > 0$, there exist $\gamma, n_0 > 0$ for which the following holds. Every graph $G = (V, E)$ with $n \ge n_0$ vertices, $m \le Cn$ edges, maximum degree at most γn, and τ tight components, admits a bisection $V = V_1 \cup V_2$ where each V_i induces at most*

$$\frac{m}{4} - \frac{n - \tau}{8} + \varepsilon n$$

edges.

Proof. Let us assume that $\varepsilon \le \frac{1}{2}$ and $C \ge 1$. (Otherwise, we may reduce ε to $\frac{1}{2}$ or increase C to 1, follow the proof below, and achieve a stronger result.) Let $\gamma = \frac{\varepsilon^4}{1024C^3}$. We will implicitly assume large n in the remainder of our argument. As in the proof of Theorem 1.3, start by taking a maximum matching $\{e_1, \ldots, e_s\}$, breaking ties by counting the number of free vertices in $W = V \setminus \{e_1, \ldots, e_s\} = \{w_1, \ldots, w_r\}$. Partition W into $W_1 \cup \cdots \cup W_s \cup T$ by placing all non-free vertices and vertices with degree $\ge \frac{C}{\varepsilon}$ into T, and for each remaining $w \in W$, place it in the lowest-indexed W_i for which w has a free-neighbor in e_i. For each i, let $U_i = e_i \cup W_i$. Since there are exactly τ non-free vertices by Lemma 3.2, and the sum of all degrees is at most $2Cn$, the combined size of T is at most

$$|T| \le \tau + 2\varepsilon n. \tag{7}$$

By our matching's maximality, no two vertices $w, w' \in W_i$ can be adjacent to different endpoints of e_i. So, all $w \in W_i$ are only adjacent to the same vertex $v_i \in e_i$, and we conclude that each U_i induces a star with apex v_i. We now randomly construct a bipartition $V = V_1 \cup V_2$ (not yet a bisection) by independently placing each v_i on a uniformly random side, and then placing the rest of $U_i \setminus \{v_i\}$ on the other side. Independently place each vertex of T on a uniformly random side. Define the random variables Y_1 and Y_2 to be the numbers of edges induced by each V_i.

The outline of the remainder of our proof is as follows.

Step 1. The expectations $\mathbb{E}[Y_1]$ and $\mathbb{E}[Y_2]$ are both at most $\frac{m}{4} - \frac{n-\tau}{8} + \frac{\varepsilon n}{4}$.

Step 2. By applying the Hoeffding-Azuma concentration inequality (Theorem 4.1) with suitable Lipschitz conditions, we find that with probability strictly greater than $\frac{2}{3}$, we have $Y_1 \leq \frac{m}{4} - \frac{n-\tau}{8} + \frac{\varepsilon n}{2}$. By symmetry, the same probabilistic bound holds for Y_2.

Step 3. Another application of Hoeffding-Azuma shows that with probability greater than $\frac{2}{3}$, the sizes of V_1 and V_2 are both within $\frac{\varepsilon n}{16C}$ of $\frac{n}{2}$.

Step 4. There must therefore be an outcome in which all above properties hold simultaneously. We now equalize the sizes of V_1 and V_2 by moving at most $\frac{\varepsilon n}{16C}$ vertices from the larger side to the smaller, and show that this only changes each $e(V_i)$ by at most $\frac{\varepsilon n}{2}$.

Step 1. We need to bound $\mathbb{E}[Y_1]$ and establish the Lipschitz condition. By symmetry, note that $\mathbb{E}[Y_1] = \mathbb{E}[Y_2]$, and so

$$\mathbb{E}[Y_1] = \frac{1}{2}\left(m - \mathbb{E}[e(V_1, V_2)]\right) . \tag{8}$$

We calculate $e(V_1, V_2)$ using the accounting procedure introduced at the beginning of Section 3. That is, we let X_i be the number of crossing edges induced by the splits of $U_1 \cup \cdots \cup U_i$, and estimate each $X_i - X_{i-1}$. This is the number of new crossing edges that are determined after splitting U_i. To estimate $\mathbb{E}[X_i - X_{i-1}]$, let k be the number of edges from U_i to previous U_j.

By symmetry, the expected number of crossing edges from U_i to earlier U_j is exactly $k/2$. The gain comes from the fact that all of the $|U_i| - 1$ edges in the star induced by U_i always become crossing edges by construction. Putting all of these calculations together, we find that

$$\begin{aligned}\mathbb{E}[X_i - X_{i-1}] &= \frac{k}{2} + \frac{|U_i| - 1}{2} + \frac{|U_i| - 1}{2} \\ &= \frac{1}{2}\left[e(U_1 \cup \cdots \cup U_i) - e(U_1 \cup \cdots \cup U_{i-1})\right] + \frac{|U_i| - 1}{2} .\end{aligned}$$

Finally, since each vertex of T is independently assigned at the end, on expectation exactly half of the edges incident to T become crossing edges. Therefore,

$$\mathbb{E}[e(V_1, V_2)] = \frac{m}{2} + \sum_{i=1}^{s} \frac{|U_i| - 1}{2} = \frac{m}{2} + \frac{n - |T| - s}{2}$$

We can bound $s \leq \frac{n-|T|}{2}$ because each U_i has at least 2 vertices (from e_i). Hence, using (7) to control $|T|$, we obtain

$$\mathbb{E}[e(V_1, V_2)] \geq \frac{m}{2} + \frac{n - |T| - \frac{n-|T|}{2}}{2} \geq \frac{m}{2} + \frac{n - (\tau + 2\varepsilon n)}{4} ,$$

which together with (8) yields

$$\mathbb{E}[Y_1] \leq \frac{1}{2}\left(m - \left(\frac{m}{2} + \frac{n-\tau-2\varepsilon n}{4}\right)\right) = \frac{m}{4} - \frac{n-\tau}{8} + \frac{\varepsilon n}{4},$$

as desired.

Step 2. For each $1 \leq i \leq s$, let C_i be the sum of the degrees of all vertices in U_i. Clearly, flipping the assignment of v_i cannot affect Y_1 by more than C_i. Also, flipping the assignment of any $w \in T$ cannot change Y_1 by more than the degree $d(w)$ of w. Therefore, if we define

$$L = \sum_{i=1}^{s}\left(\sum_{u \in U_i} d(u)\right)^2 + \sum_{w \in T} d(w)^2,$$

the Hoeffding-Azuma inequality (Theorem 4.1) gives

$$\mathbb{P}\left(Y_1 \geq \mathbb{E}[Y_1] + \frac{\varepsilon n}{4}\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{32L}\right\}. \tag{9}$$

Let us now control L. We observed in Step 1 that each U_i induced a star with apex $v_i \in e_i$. Let u_i be the other endpoint of e_i. This gives $|U_i| \leq d(v_i) + 1$, and by construction, every other vertex in U_i has degree less than $\frac{C}{\varepsilon}$. So,

$$\sum_{u \in U_i} d(u) \leq d(v_i) + d(u_i) + (d(v_i) - 1)\frac{C}{\varepsilon} \leq (d(u_i) + d(v_i))\frac{2C}{\varepsilon},$$

and hence

$$\begin{aligned}
L &\leq \frac{4C^2}{\varepsilon^2}\sum_{i=1}^{s}(d(u_i) + d(v_i))^2 + \sum_{w \in T} d(w)^2 \\
&\leq \frac{8C^2}{\varepsilon^2}\sum_{i=1}^{s}(d(u_i)^2 + d(v_i)^2) + \sum_{w \in T} d(w)^2 \\
&\leq \frac{8C^2}{\varepsilon^2}\sum_{v \in V} d(v)^2 \leq \frac{8C^2}{\varepsilon^2}(\gamma n)\sum_{v \in V} d(v) \\
&\leq \frac{8C^2}{\varepsilon^2}(\gamma n)(2Cn),
\end{aligned}$$

where we used that all $d(v) \leq \gamma n$ and that sum of all degrees in G is at most $2Cn$. However, we chose $\gamma = \frac{\varepsilon^4}{1024C^3}$, so we have $L \leq \frac{\varepsilon^2 n^2}{64}$. Substituting this into (9), we conclude that

$$\mathbb{P}\left(Y_1 \geq \mathbb{E}[Y_1] + \frac{\varepsilon n}{4}\right) \leq 2e^{-2} < \frac{1}{3},$$

as desired.

Step 3. By symmetry, the expected size of V_1 is exactly $\frac{n}{2}$. Switching the choice of a single v_i can only change $|V_1|$ by at most $|U_i| - 2 \leq d(v_i) - 1$, and switching the choice of a single $w \in T$ can only change $|V_1|$ by at most 1. Therefore, if we define

$$L' = \sum_{i=1}^{s}(d(v_i) - 1)^2 + |T|,$$

the Hoeffding-Azuma inequality (Theorem 4.1) gives

$$\mathbb{P}\left(\left||V_1| - \frac{n}{2}\right| \geq \frac{\varepsilon n}{16C}\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{512C^2 L'}\right\}. \tag{10}$$

As in Step 2, we may control L' by

$$L' \leq |T| + \sum_{i=1}^{s} d(v_i)^2 \leq n + (\gamma n)\sum_{i=1}^{s} d(v_i) \leq n + (\gamma n)(2Cn) = (1+o(1))\frac{\varepsilon^4 n^2}{512C^2}$$

Substituting this into (10), we find that

$$\mathbb{P}\left(\left||V_1| - \frac{n}{2}\right| \geq \frac{\varepsilon n}{16C}\right) \leq 2e^{-(1+o(1))/\varepsilon^2} < \frac{1}{3}.$$

Since $|V_1|+|V_2| = n$, whenever $|V_1|$ is within $\frac{\varepsilon n}{16C}$ of $\frac{n}{2}$, $|V_2|$ is as well. Hence with probability strictly greater than $\frac{2}{3}$, both $|V_1|$ and $|V_2|$ are within $\frac{\varepsilon n}{16C}$ of $\frac{n}{2}$.

Step 4. Our probabilistic arguments show that there is a partition $V = V_1 \cup V_2$ with the properties that each $|V_i|$ is within $\frac{\varepsilon n}{16C}$ of $\frac{n}{2}$, and each $e(V_i)$ is at most $\frac{m}{4} - \frac{n-\tau}{8} + \frac{\varepsilon n}{2}$. To equalize the sizes of $|V_1|$ and $|V_2|$, we now move the lowest-degree vertices from the larger side to the smaller.

Since the sum of all degrees is at most $2Cn$, the number of vertices whose degree exceeds $8C$ is at most $\frac{n}{4}$. In particular, we may equalize $|V_1|$ and $|V_2|$ by moving vertices of degree below $8C$. This affects each $e(V_i)$ by at most $8C \cdot \frac{\varepsilon n}{16C} = \frac{\varepsilon n}{2}$, thereby producing the desired result. □

4.2 Judicious bisection with initial pre-partition

Theorem 1.11 produces a judicious bisection which in fact surpasses the expected performance of the elementary random bisection algorithm, when degrees are bounded by $o(n)$ and there are only $O(n)$ edges. In Section 2, we showed that when there are $\omega(n)$ edges, then the elementary random bisection algorithm already achieves the bound of Theorem 1.9. So, the main obstacle is the maximum degree condition.

To work around this, we will often pre-allocate vertices of high degree into V_1 and V_2, and then apply a randomized algorithm to distribute the remaining vertices. The same proof as we used in the previous section will also produce the following corollary.

Corollary 4.2. *Given any real constants $C, \varepsilon > 0$, there exist $\gamma, n_0 > 0$ for which the following holds. Let $G = (V, E)$ be a given graph with $n \geq n_0$ vertices and at most Cn edges, and let $A \subset V$ be a set of $\leq \gamma n$ vertices which have already been partitioned into $A_1 \cup A_2$. Let $\overline{A} = V \setminus A$, and suppose that every vertex in $\overline{A}$ has degree at most γn (with respect to the full G). Let τ be the number of tight components in $G[\overline{A}]$. Then, there is a bisection $V = V_1 \cup V_2$ with $A_1 \subset V_1$ and $A_2 \subset V_2$, such that both*

$$e(V_1) \leq e(A_1) + \frac{e(A_1, \overline{A})}{2} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \varepsilon n$$
$$e(V_2) \leq e(A_2) + \frac{e(A_2, \overline{A})}{2} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \varepsilon n.$$

Proof. Again, assume that $\varepsilon \le \frac{1}{2}$ and $C \ge 1$. Let $\gamma = \frac{\varepsilon^4}{1024C^3}$. We will implicitly assume large n in the remainder of our argument. As in the proof of Theorem 1.11, start by taking a maximum matching $\{e_1, \ldots, e_s\}$ in $G[\overline{A}]$, breaking ties by counting the number of free vertices in $W = \overline{A} \setminus \{e_1, \ldots, e_s\} = \{w_1, \ldots, w_r\}$. Partition W into $W_1 \cup \cdots \cup W_s \cup T$ by placing all non-free vertices and vertices with degree $\ge \frac{C}{\varepsilon}$ (with respect to the full G) into T, and for each remaining $w \in W$, place it in the lowest-indexed W_i for which w has a free-neighbor in e_i. For each i, let $U_i = e_i \cup W_i$. Note that Lemma 3.2 bounds the number of non-free vertices by the number of tight components τ, and since the sum of all degrees is at most $2Cn$, the combined size of T is at most

$$|T| \le \tau + 2\varepsilon n\,.$$

As before, each U_i induces a star with apex v_i. Randomly construct a bipartition by independently placing each v_i on a uniformly random side, and then placing the rest of $U_i \setminus \{v_i\}$ on the other side. Independently place each vertex of T on a uniformly random side. Define the random variables Y_1 and Y_2 to be the numbers of edges induced by each V_i. A similar argument to that of Step 1 before now establishes that

$$\mathbb{E}[Y_1] = e(A_1) + \frac{e(A_1, \overline{A})}{2} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \frac{\varepsilon n}{4}$$
$$\mathbb{E}[Y_2] = e(A_2) + \frac{e(A_2, \overline{A})}{2} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \frac{\varepsilon n}{4}\,.$$

The same considerations as in Step 2 before then establish that each

$$\mathbb{P}\left(Y_i \ge \mathbb{E}[Y_i] + \frac{\varepsilon n}{4}\right) \le 2e^{-2} < \frac{1}{3}$$
$$\mathbb{P}\left(\left||V_1 \setminus A_1| - \frac{|\overline{A}|}{2}\right| \ge \frac{\varepsilon n}{16\sqrt{2C}}\right) \le 2e^{-1/(2\varepsilon^2)} \le 2e^{-2} < \frac{1}{3}\,,$$

so there is an outcome where each $|V_i \setminus A_i|$ deviates from its mean by at most $\frac{\varepsilon n}{16\sqrt{2C}}$, and each

$$e(V_i) \le e(A_i) + \frac{e(A_i, \overline{A})}{2} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \frac{\varepsilon n}{2}\,.$$

Since $|A| \le \gamma n$, we can now equalize the sizes of V_1 and V_2 by moving fewer than $\frac{\varepsilon n}{16\sqrt{2C}} + |A| \le \frac{\varepsilon n}{16C}$ of the lowest-degree vertices from the larger side to the smaller side. The same argument as in Step 4 before then completes the proof. □

5 Minimum degree two

Corollary 4.2, our final result from the previous section, now enables us to prove Theorem 1.9, which achieves the asymptotically optimal bisection in graphs with prescribed minimum degree δ. Since the arguments become increasingly complicated as δ gets larger, we start with the $\delta = 2$ case. The proof of this simple case already contains (in a more simple form) most of the main ideas for the proof of the general case, and thus we hope that its proof serves as an illustrative starting point. In this section, we prove the following result.

Theorem. *For every $\varepsilon > 0$, there exists n_0 such that every graph $G = (V, E)$ with $n > n_0$ vertices and minimum degree at least 2 has a bisection $V = V_1 \cup V_2$ in which each $e(V_i) \le \left(\frac{1}{3} + \varepsilon\right)e(G)$.*

Proof. Let m be the number of edges in G. By Theorem 1.7 (with $\varepsilon = \frac{1}{12}$), and by the final argument in its proof, we may assume that $\frac{n}{4} \le m \le 144n$. This observation translates into $m = \Theta(n)$, where the notation has masked absolute constants $\frac{1}{4}$ and 144.

If all degrees are below $n^{3/4}$, then we are already done by Theorem 1.11. Otherwise, let A be the set of vertices with degree at least $n^{3/4}$ and let $\overline{A} = V \setminus A$. Since $m = \Theta(n)$, we have $|A| = O(n^{1/4})$, and hence $e(A) = o(m)$. Let $m_1 = e(A, \overline{A})$ and $m_2 = e(\overline{A})$, and note that $m = m_1 + m_2 + o(m)$. Since $e(A) = o(m)$, we will essentially ignore the edges inside A. Let Δ be the maximum $\overline{A}$-degree in A, i.e., the maximum number of edges to $\overline{A}$ that any vertex of A has.

Start the bipartition $V = V_1 \cup V_2$ by splitting $A = A_1 \cup A_2$ in such a way that $e(A_1, \overline{A}) \ge e(A_2, \overline{A})$, but $e(A_1, \overline{A})$ and $e(A_2, \overline{A})$ are as close as possible. By the optimality of the splitting, we have $e(A_1, \overline{A}) - e(A_2, \overline{A}) \le \Delta$, since otherwise we can improve this partition by moving one vertex from A_1 to A_2. Note that if this achieves $\frac{m_1}{3} \le e(A_1, \overline{A}) \le \frac{2m_1}{3}$, then $e(A_2, \overline{A})$ also satisfies the same bound, and Corollary 4.2 applied with $\tau \le n$ then produces a bisection with the desired

$$e(V_i) \le e(A) + \frac{2m_1/3}{2} + \frac{m_2}{4} + o(m) \le \frac{m}{3} + o(m)$$

Therefore, we may assume that the vertices of A cannot be split as evenly as above. Yet if $\frac{3}{2}\Delta \le m_1 \le 3\Delta$, then placing the vertex with Δ on one side, and the rest of A on the other already gives $\frac{m_1}{3} \le e(A_1, \overline{A}) \le \frac{2m_1}{3}$. Also, if $m_1 > 3\Delta$, then the optimal split of A will give

$$\frac{m_1}{2} \le d(A_1, \overline{A}) \le \frac{m_1}{2} + \frac{\Delta}{2} < \frac{2m_1}{3}\,.$$

Consequently, it remains to study the situation when $\Delta \le m_1 < \frac{3}{2}\Delta$. The optimal split of A then consists of putting the Δ-vertex alone in A_1, and placing all other vertices of A into A_2.

Case 1: $\boldsymbol{m_2 \ge 6\Delta - 4m_1}$. We now have $e(A_1, \overline{A}) = \Delta$ and $e(A_2, \overline{A}) = m_1 - \Delta < \frac{\Delta}{2}$. Corollary 4.2 applied with $\tau \le n$ then produces a bisection with the desired

$$e(V_i) \le e(A) + \frac{\Delta}{2} + \frac{m_2}{4} + o(m) \le \frac{4m_1 + m_2}{12} + \frac{m_2}{4} + o(m) = \frac{m}{3} + o(m)\,.$$

Case 2: $\boldsymbol{m_2 < 6\Delta - 4m_1}$. To settle this case, we will first bound the number of tight components in terms of n and m_2. Let $n' = |\overline{A}|$, and let τ_1 be the number of isolated vertices in $G[\overline{A}]$. Since A_1 consists of a single vertex, there are exactly $m_1 - \Delta$ edges from A_2 to $\overline{A}$. The graph G had minimum degree at least 2, so every isolated vertex in $G[\overline{A}]$ consumes at least one edge from A_2 (it can only lose at most one edge to the lone vertex in A_1). Therefore, $\tau_1 \le m_1 - \Delta$.

Let t be the maximum number of vertex-disjoint triangles in $G[\overline{A}]$. Since $G[\overline{A}]$ has τ_1 isolated vertices, we immediately have $m_2 \ge 3t + \frac{n' - \tau_1 - 3t}{2}$, or equivalently, $3t \le 2m_2 - n' + \tau_1$. Every tight component of $G[\overline{A}]$ either is an isolated vertex, or contains a triangle, so we may bound the number of tight components τ by

$$\tau \le \tau_1 + t \le \tau_1 + \frac{2m_2 - n' + \tau_1}{3} = \frac{4\tau_1 + 2m_2 - n'}{3}\,.$$

By construction, all degrees in $G[\overline{A}]$ are below $n^{3/4}$, so Theorem 1.3 produces a bisection $\overline{A} = B_1 \cup B_2$ of $G[\overline{A}]$ of size at least

$$\frac{m_2}{2} + \frac{n' - \tau - n^{3/4}}{4} \ge o(m) + \frac{m_2}{2} + \frac{n'}{4} - \frac{4\tau_1 + 2m_2 - n'}{12} = o(m) + \frac{m_2 + n' - \tau_1}{3}\,,$$

which by $\tau_1 \le m_1 - \Delta$ is at least

$$e(B_1, B_2) \ge o(m) + \frac{m_2 + n' - m_1 + \Delta}{3}\,.$$

Without loss of generality, suppose that $e(A_1, B_1) + e(B_1) \le e(A_1, B_2) + e(B_2)$. (Otherwise, we may swap B_1 and B_2.) Let $V_i = A_i \cup B_i$. Then, we have

$$\begin{aligned} e(V_1) = e(A_1) + e(A_1, B_1) + e(B_1) &\le e(A_1) + \frac{e(A_1, B_1) + e(A_1, B_2) + e(B_1) + e(B_2)}{2} \\ &= o(m) + \frac{\Delta + (m_2 - e(B_1, B_2))}{2} \\ &\le o(m) + \frac{\Delta + m_2}{2} - \frac{m_2 + n' - m_1 + \Delta}{6} \\ &= o(m) + \frac{\Delta}{3} + \frac{m_2}{3} - \frac{n'}{6} + \frac{m_1}{6}\,. \end{aligned}$$

Using $n' \ge \Delta$, $\Delta \le m_1$, and $m = m_1 + m_2 + o(m)$, this is at most

$$e(V_1) \le o(m) + \frac{\Delta}{6} + \frac{m_2}{3} + \frac{m_1}{6} \le o(m) + \frac{m_1}{3} + \frac{m_2}{3} = o(m) + \frac{m}{3}\,.$$

On the other hand,

$$\begin{aligned} e(V_2) = e(A_2) + e(A_2, B_2) + e(B_2) &\le o(m) + e(A_2, \overline{A}) + \big(e(\overline{A}) - e(B_1, B_2)\big) \\ &\le o(m) + (m_1 - \Delta) + \left(m_2 - \frac{m_2 + n' - m_1 + \Delta}{3}\right) \\ &= o(m) + \frac{4m_1 - 4\Delta + 2m_2 - n'}{3} \\ &= o(m) + \frac{m_1 + m_2}{3} + \frac{3m_1 - 4\Delta + m_2 - n'}{3}\,. \end{aligned}$$

Using our assumption that $m_2 < 6\Delta - 4m_1$ in this case, this is at most

$$e(V_2) \le o(m) + \frac{m}{3} + \frac{2\Delta - m_1 - n'}{3}\,.$$

Since $\Delta \le m_1$ and $\Delta \le n' + o(n)$, we conclude that $e(V_2) \le o(m) + \frac{m}{3}$, as desired.

Therefore, our partition $V = V_1 \cup V_2$ has both $e(V_i) \le \frac{m}{3} + o(m)$. Also, $|V_1|$ and $|V_2|$ differ by at most $|A| = O(n^{1/4})$. Since $m \le 144n$, at most $\frac{n}{2}$ vertices of G can have degree over 576. Hence, there are at least $\frac{n}{2}$ vertices with degree at most 576. We equalize $|V_1|$ and $|V_2|$ by moving $O(n^{1/4})$ such vertices across. This affects each $e(V_i)$ by under $O(n^{1/4}) = o(m)$, thereby completing the proof. □

6 Minimum degree: general case

In this section, we establish the general case of our main theorem, which we restate below for the reader's convenience.

Theorem. *Given any $\varepsilon > 0$ and any positive even integer δ, there exists n_0 such that every graph $G = (V, E)$ with $n \geq n_0$ vertices, m edges, and minimum degree at least δ, has a bisection in which for both $i = 1, 2$,*

$$e(V_i) \leq \left(\frac{\delta+2}{4(\delta+1)} + \varepsilon \right) m \,.$$

Proof. By Theorem 1.7 and the final argument in its proof, we may assume that $\frac{n}{4} \leq m \leq \varepsilon^{-2} n$. Throughout this proof, ε will be fixed, and it will be convenient to employ asymptotic notation which implicitly assumes that ε is a constant. For example, our initial observation translates into $m = \Theta(n)$.

We first describe the idea of the remainder of the proof, which is similar to that of the case $\delta = 2$ given in the previous section. Start by identifying the set of vertices A which have large degree (degree at least $n^{3/4}$), and let $A_1 \cup A_2$ be a bipartition of these vertices so that $e(A_1, V \setminus A)$ and $e(A_2, V \setminus A)$ are as close as possible. If the gap between these two is small enough, then a naive random split of the remaining vertices will give the result we were hoping for. Otherwise, we would like to make use of Corollary 4.2. Most of the technical details of our proof lie in this step, or more precisely, in estimating the number of tight components in $G[V \setminus A]$. Recall that to do this in the $\delta = 2$ case, we first deduced a structural information about the degree of the vertices in A (there we knew that A_1 consisted of a single vertex), and then used this information to effectively bound the number of tight components. The general idea is the same here, but things get more complicated because both the structural information, and the method of bounding tight components under this information becomes a lot more difficult (and computationally involved).

We now give the details of the proof. If all degrees are below $n^{3/4}$, then we are already done by Theorem 1.11. Otherwise, let A be the set of vertices with degree at least $n^{3/4}$ and let $\overline{A} = V \setminus A$ (we call the vertices in A as the *large degree* vertices). Since $m = \Theta(n)$, we have $|A| = O(n^{1/4})$, and hence $e(A) = o(m)$. Let

$$n' = |\overline{A}| \,, \qquad m_1 = e(A, \overline{A}) \,, \quad \text{and} \quad m_2 = e(\overline{A}) \,,$$

and note that

$$n' = n - o(n) \quad \text{and} \quad m = m_1 + m_2 + o(m) \,. \tag{11}$$

Start the bipartition $V = V_1 \cup V_2$ by splitting $A = A_1 \cup A_2$ in such a way that $e(A_1, \overline{A}) \geq e(A_2, \overline{A})$, but $e(A_1, \overline{A})$ and $e(A_2, \overline{A})$ are as close as possible. Define the parameter

$$\theta = e(A_1, \overline{A}) - e(A_2, \overline{A}) \,,$$

which quantifies the "gap" in the optimal split. By construction, we now have

$$e(A_2, \overline{A}) \leq e(A_1, \overline{A}) = \frac{m_1 + \theta}{2} \,,$$

so Corollary 4.2 produces a bisection in which both

$$e(V_i) \leq \frac{m_1 + \theta}{4} + \frac{m_2}{4} - \frac{n - \tau}{8} + o(m) \,,$$

where τ is the number of tight components in $G[\overline{A}]$. Using (11) to combine m_1 and m_2, we observe that it therefore suffices to show that

$$\frac{m}{4} + \frac{\theta}{4} - \frac{n - \tau}{8} \leq \frac{\delta + 2}{4(\delta + 1)} m \,,$$

or equivalently,

$$\theta + \frac{\tau}{2} \le \frac{n}{2} + \frac{m}{\delta+1}\,. \tag{12}$$

Since we always have $\tau \le n$, we see that we are already done if $\theta \le \frac{m}{\delta+1}$. (It is worth noting that $\theta = \frac{m}{\delta+1}$ corresponds to the extremal example $K_{\delta+1,n-\delta-1}$, as there $m \approx (\delta+1)n$, there are exactly $\delta + 1$ large degree vertices, and the optimal split puts $\frac{\delta}{2} + 1$ of them in A_1 and the other $\frac{\delta}{2}$ in A_2.) Therefore, we will assume for the remainder of the proof that the "gap" between $e(A_1, \overline{A})$ and $e(A_2, \overline{A})$ is rather large:

$$\theta > \frac{m}{\delta+1}\,. \tag{13}$$

In fact, this gap is so wide that it has serious implications regarding the nature of the $A = A_1 \cup A_2$ split. Recall that in the $\delta = 2$ case (previous section), we deduced that A_1 only contained a single vertex of degree Δ, and $e(A_2, \overline{A})$ was below $\frac{\Delta}{2}$. The following result, which we prove in Section 6.1, is its analogue in the general case. Remarkably, its proof only uses the optimality of the $A = A_1 \cup A_2$ split. In order to maintain continuity in our exposition, we postpone the proof of this lemma to Section 6.1 below.

Lemma 6.1. *If $\theta > \frac{m}{\delta+1}$, then in the optimal split $A = A_1 \cup A_2$:*

(i) *All $\overline{A}$-degrees in A_1 are at least θ.*

(ii) *Only at most $\delta - 1$ vertices in $A_1 \cup A_2$ have $\overline{A}$-degree at least θ.*

(iii) *The sum of the $\overline{A}$-degrees over all other vertices (in A_2) is at most $n' - \theta$.*

Remark. We now also have $\theta \le n'$, because the $\overline{A}$-degree of the vertex in A_1 is at least θ, but cannot exceed $|\overline{A}| = n'$. We refer to the vertices of $\overline{A}$-degree at least θ as the *huge $\overline{A}$-degree vertices* or simply *huge degree vertices.*

Let α be the number of vertices in $A_1 \cup A_2$ with $\overline{A}$-degree at least θ, and let ρ be the sum of the $\overline{A}$-degrees of all other vertices (which are all in A_2). Note that the sum of the $\overline{A}$-degrees of the vertices in A is then

$$m_1 \ge \alpha\theta + \rho\,. \tag{14}$$

Returning to verify (12), we will need to control the number of tight components. In the simple case when $\delta = 2$, we were able to do this by counting the number of isolated vertices, and the number of vertex-disjoint triangles. However, things get more difficult to control for general δ, and the most efficient approach is to bound τ in terms of the degree sequence of $G[\overline{A}]$, via the following result, which essentially claims that the optimum is achieved when all vertices are packed into tight components that are complete graphs of odd order.

Lemma 6.2. *Let H be an arbitrary graph. For each integer i, let d_i be the number of vertices in H with degree equal to i. Then the number of tight components τ in H satisfies*

$$\tau \le \frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots$$

Remark. Odd-degree vertices cannot contribute to tight components because for every vertex v in a tight component T, there is a perfect matching of $T \setminus \{v\}$, and v has either 0 or 2 neighbors in each edge of the matching. This is why the bound above does not involve d_1, d_3, d_5, etc.

The degree sequence does not appear in (12), however, and there are two ways to proceed and relate the degree sequence with other parameters. The simpler approach crudely controls the degree sequence in terms of α and m_1 (or more precisely α and ρ), using the minimum degree condition of G. (For example, if m_1 were 0, then the minimum degree condition would force $d_0 = d_1 = \cdots = d_{\delta-1} = 0$.) The more intricate approach adds an additional information by relating the degree sequence with m_2 as well.

We will actually take both approaches by the end of this proof, but it turns out to be more convenient to use the crude approach first, because it will already work for most of the (non-tight) cases. Applying the minimum degree condition in G to the sum of all degrees in $\overline{A}$, we find that

$$\delta n' \leq 2m_2 + m_1 \, ,$$

which means that

$$m \geq m_1 + m_2 \geq \frac{\delta n' + m_1}{2} \, . \tag{15}$$

We then make use of the fact that A contains α huge degree vertices. Note that even if all the α vertices in A had full degree to $\overline{A}$, the only way that a vertex in $G[\overline{A}]$ could have degree below $\delta - \alpha$ would be if it was incident to edges counted by ρ. Consequently, we get the following constraint:

$$(\delta - \alpha)d_0 + (\delta - \alpha - 1)d_1 + \cdots + d_{\delta-\alpha-1} \leq \rho \, . \tag{16}$$

Then, it turns out that the most economical way to control the bound in Lemma 6.2 is to use ρ to reduce as many degrees to 0 as possible. This produces the following result, whose complete proof is in Section 6.3.

Lemma 6.3. *The induced subgraph $G[\overline{A}]$ has at most $\frac{n'+\rho}{\delta-\alpha+1}$ tight components.*

By (15) and (14), we get $m \geq \frac{\delta n' + \alpha\theta + \rho}{2}$. Using this bound and Lemma 6.3 in (12), we see that it suffices to show

$$\theta + \frac{n' + \rho}{2(\delta - \alpha + 1)} \leq \frac{n}{2} + \frac{\delta n' + \alpha\theta + \rho}{2(\delta + 1)} \, ,$$

or since $n' \leq n$, that

$$\theta + \frac{n' + \rho}{2(\delta - \alpha + 1)} \leq \frac{2\delta + 1}{2(\delta + 1)} n' + \frac{\alpha\theta + \rho}{2(\delta + 1)} \, . \tag{17}$$

It turns out that the above inequality is true for most values of α and δ (for the remaining cases, we will use the slightly different approach that we briefly discussed above). Note that by Lemma 6.1(ii), we have $1 \leq \alpha \leq \delta - 1$. It turns out that the $\alpha = 1$ case is the most delicate, as that corresponds to an extremal construction and cannot be resolved by our current approach. The $\alpha = 2$ case will be easily proved using Lemma 6.3 and (12). For $3 \leq \alpha \leq \delta - 1$, observe that the left hand side of (17) is convex in α in that range, while the right hand side is linear in α. Therefore, once we verify it for $\alpha = 3$ and $\alpha = \delta - 1$, all intermediate values of α will follow by convexity.

Lemma 6.4. *Let $\alpha = 2$, and $\delta \geq 4$ be an even integer. Then* (12) *holds.*

Lemma 6.5. *Let $\delta \geq 6$ be an even integer, and let n', θ, ρ be non-negative real numbers satisfying*

$$\theta \leq n', \qquad \rho \leq n' - \theta.$$

Then (17) *holds when $\alpha = 3$, and when $\alpha = \delta - 1$.*

Apart from the most delicate $\alpha = 1$ case, only the case $(\delta, \alpha) = (4, 3)$ remains. For both of these remaining situations, we now take the refined approach alluded to earlier, where we employ the entire degree sequence of $G[\overline{A}]$. Since $e(A) = o(m)$, the total number of edges satisfies

$$m \geq \alpha\theta + \rho + \frac{1}{2}(d_1 + 2d_2 + 3d_3 + \cdots) + o(m), \tag{18}$$

where d_i is the number of vertices of degree i in the induced subgraph $G[\overline{A}]$.

Using this inequality and Lemma 6.2, we see that to prove (12), it suffices to show

$$\theta + \frac{1}{2}\left(\frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots\right) \leq \frac{n'}{2} + \frac{1}{\delta+1}\left(\alpha\theta + \rho + \frac{1}{2}(d_1 + 2d_2 + 3d_3 + \cdots)\right), \tag{19}$$

when the constraint (16) on the degree sequence holds.

Lemma 6.6. *Let δ be a positive integer, and let $n', \theta, \rho, d_0, d_1, \ldots$ be non-negative real numbers satisfying*

$$\theta \leq n', \qquad \rho \leq n' - \theta, \quad \text{and} \quad \sum_i d_i = n',$$

together with (16)*. Then Inequality* (19) *holds for $(\delta, \alpha) = (4, 3)$, as well as for $\alpha = 1$ and arbitrary $\delta \geq 2$.*

This final ingredient completes our proof of Theorem 1.9, modulo the intermediate lemmas. □

6.1 Splitting large degree vertices

In this section, we prove Lemma 6.1 by establishing a stronger, but more technical result. Let us parameterize the balance between the $e(A_i, \overline{A})$ in the optimal split by defining λ such that $e(A_1, \overline{A}) = \lambda m_1$ and $e(A_2, \overline{A}) = (1 - \lambda)m_1$. Recall that $m_1 = e(A, \overline{A})$, $m_2 = e(\overline{A})$, and note that now

$$\theta = e(A_1, \overline{A}) - e(A_2, \overline{A}) = (2\lambda - 1)m_1.$$

By construction, we always have $\frac{1}{2} \leq \lambda \leq 1$. As it turns out, the particular location of λ gives useful information about the nature of the partition $A_1 \cup A_2$.

Lemma 6.7. *Let κ be an integer for which $\lambda > \frac{\kappa+1}{2\kappa+1}$. Then both:*

(i) *$|A_1| \leq \kappa$, and all vertices of A_1 have $\overline{A}$-degree at least θ; and*

(ii) *A_2 contains at most $\kappa - 1$ vertices with $\overline{A}$-degree $\geq \theta$, and the $\overline{A}$-degrees of all remaining vertices sum to at most $n' - \theta$.*

Proof. Our argument is based on the fact that no subset of A has $\overline{A}$-degree sum strictly between $(1-\lambda)m_1$ and λm_1. Let $a = |A_1|$, and suppose for contradiction that $a \ge \kappa + 1$. Then, by moving the vertex of lowest $\overline{A}$-degree from A_1 to A_2, the remaining $\overline{A}$-degree sum of A_1 would be under λm_1, but at least

$$\frac{a-1}{a} \cdot \lambda m_1 > \frac{\kappa}{\kappa+1} \cdot \frac{\kappa+1}{2\kappa+1} m_1 = \frac{\kappa}{2\kappa+1} m_1 > (1-\lambda)m_1 \,,$$

thereby contradicting the optimality of the $A_1 \cup A_2$ split. Hence we indeed have $|A_1| \le \kappa$. Furthermore, if any vertex of A_1 has $\overline{A}$-degree strictly less than θ, then we can also improve the split by moving it to A_2. This establishes (i).

For the first part of (ii), observe that if A_2 contains κ vertices with $\overline{A}$-degree at least $\theta = (2\lambda - 1)m_1$, then the $\overline{A}$-degree sum of A_2 is already at least

$$\kappa \cdot (2\lambda - 1)m_1 > \frac{\kappa}{2\kappa+1} m_1 > (1-\lambda)m_1 \,,$$

contradicting the definition of λ.

The final part of (ii) is slightly more involved. Let v be a vertex of A_1 of minimal $\overline{A}$-degree, and let that degree be d. We will show that the $\overline{A}$-degree sum of all vertices in A_2 with $\overline{A}$-degree less than $\theta = (2\lambda - 1)m_1$ is at most $d - (2\lambda - 1)m_1$, which will suffice because d is clearly at most $|\overline{A}| = n'$. Indeed, assume for contradiction that this is not the case, and consider moving v to A_2. The $\overline{A}$-degree sum of A_1 will fall to $\lambda m_1 - d$, which must be at most $(1-\lambda)m_1$ by the assumed optimality of the $A_1 \cup A_2$ split. Now consider moving the vertices with $\overline{A}$-degree below θ from A_2 to A_1, one by one. If we moved all of them, then A_1 would achieve a total $\overline{A}$-degree sum strictly greater than

$$(\lambda m_1 - d) + (d - (2\lambda - 1)m_1) = (1-\lambda)m_1 \,.$$

Yet each vertex that we move has $\overline{A}$-degree strictly less than $\theta = \lambda m_1 - (1-\lambda)m_1$, so at some point in the process, we must have a split where both $e(A_i, \overline{A})$ are strictly between $(1-\lambda)m_1$ and λm_1. This contradicts the optimality of the original $A_1 \cup A_2$ split, and completes our proof. □

We now observe that Lemma 6.1 follows easily from Lemma 6.7.

Proof of Lemma 6.1. Since $\theta = (2\lambda - 1)m_1$, our assumption that $\theta > \frac{m}{\delta+1} \ge \frac{m_1}{\delta+1}$ gives $\lambda > \frac{\delta+2}{2(\delta+1)}$. Recalling that δ is even, let $\kappa = \frac{\delta}{2}$, and observe that $\frac{\kappa+1}{2\kappa+1} = \frac{\delta+2}{2(\delta+1)}$, so we may apply Lemma 6.7 with $\kappa = \frac{\delta}{2}$, and Lemma 6.1 follows immediately. □

6.2 Tight components and the degree sequence

Consider an arbitrary graph H. For each i, let d_i denote the number of vertices in H whose degree is exactly i. In this section, we prove Lemma 6.2, which claims that the number of tight components in H is at most

$$\frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots$$

Proof. For each odd i, let t_i denote the number of tight components of H which have exactly i vertices. Tight components have odd order and only contain even-degree vertices, because for every vertex v in a tight component T, there is a perfect matching of $T \setminus \{v\}$, and v has either 0 or 2 neighbors in each edge of the matching. Hence tight components of order i are entirely composed of vertices of even degree $\leq i-1$. We therefore have the inequalities:

$$\begin{aligned} t_1 &\leq d_0 \\ t_1 + 3t_3 &\leq d_0 + d_2 \\ t_1 + 3t_3 + 5t_5 &\leq d_0 + d_2 + d_4 \\ &\vdots \end{aligned}$$

We claim that even when the t_i are allowed to take real values (as opposed to integers), the maximum value of $t_1 + t_3 + \cdots$ is achieved by the greedy algorithm which chooses $t_1 = \frac{d_0}{1}$, $t_3 = \frac{d_2}{3}$, $t_5 = \frac{d_4}{5}$, etc. This is the unique solution when all of the inequalities are tight.

Indeed, assume for contradiction that we have an optimal real solution in which the inequality $t_1+3t_3+\cdots+it_i \leq d_0+d_2+\cdots+d_{i-1}$ is not tight. Then, one can improve the solution by increasing t_i by some $(i+2)\varepsilon$ and decreasing t_{i+2} by $i\varepsilon$. Since this perturbation keeps $t_1+3t_3+\cdots+(i+2)t_{i+2}$ constant, it is clearly still a feasible solution, but it increases the objective $t_1 + t_3 + t_5 + \cdots$ by $2\varepsilon > 0$, contradicting our assumed optimality.

Therefore, the number of tight components $t_1 + t_3 + t_5 + \cdots$ is at most $\frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots$, as claimed. □

6.3 Tight components and the minimum degree condition

In this section, we prove Lemma 6.3, which claims that the number of tight components in $G[\overline{A}]$ is at most $\frac{n'+\rho}{\delta-\alpha+1}$. Our main tool is Lemma 6.2, which implies the following weaker bound on τ in terms of the degree sequence of $G[\overline{A}]$:

$$\tau \leq \frac{d_0}{1} + \frac{d_1}{2} + \frac{d_2}{3} + \cdots$$

(Recall that d_i is the number of vertices of $G[\overline{A}]$ that have degree exactly i.)

To get an effective bound on τ, we would like to see how large the right hand side of the above can be. Our constraints are (16), which we restate here,

$$(\delta - \alpha)d_0 + (\delta - \alpha - 1)d_1 + \cdots + d_{\delta-\alpha-1} \leq \rho\,, \tag{20}$$

together with the following obvious constraint (which we relax from equality)

$$\sum_i d_i \leq n'\,. \tag{21}$$

We claim that when the d_i are allowed to take any non-negative real values (as opposed to integers), the maximum value of $\frac{d_0}{1} + \frac{d_1}{2} + \cdots$ subject to the constraints (20) and (21) is achieved when d_0 and $d_{\delta-\alpha}$ are the only nonzero d_i.

To see this, consider an optimizer, and note that if any $i > \delta - \alpha$ has $d_i > 0$, then we can maintain the left hand sides of (20) and (21) by reducing d_i to 0, and increasing $d_{\delta-\alpha}$ by the same amount. This increases the value of the objective. Next observe that if any $0 < i < \delta-\alpha$ has $d_i > 0$, then we can maintain the left hand side of (20) by reducing d_i to 0, and increasing d_0 by $\frac{\delta-\alpha-i}{\delta-\alpha}d_i$.

This only reduces the left hand side of (21), but increases the objective by $\frac{\delta-\alpha-i}{\delta-\alpha}d_i - \frac{d_i}{i+1}$. Since $\frac{\delta-\alpha-i}{\delta-\alpha} \geq \frac{1}{i+1}$ for all $0 \leq i \leq \delta-\alpha-1$, we see that this cannot decrease the objective.

Therefore, we may assume that our optimizer is only supported at d_0 and $d_{\delta-\alpha}$. Inequality (16) then simplifies to $d_0 \leq \frac{\rho}{\delta-\alpha}$. In the objective, the coefficient of d_0 is strictly greater than that of d_δ, hence to maximize the objective, we must choose d_0 to be as large as possible, i.e., $d_0 = \frac{\rho}{\delta-\alpha}$, and then choose maximum $d_{\delta-\alpha}$ subject to the constraint (21), which says $d_0 + d_{\delta-\alpha} \leq n'$. Therefore, the objective is at most

$$\frac{\rho/(\delta-\alpha)}{1} + \frac{n' - (\rho/(\delta-\alpha))}{\delta-\alpha+1} = \frac{n'+\rho}{\delta-\alpha+1}\,,$$

as claimed. □

6.4 Two huge degree vertices

In this section, we prove Lemma 6.4, which considers the case when A contains exactly two huge degree vertices (vertices with $\overline{A}$-degree at least θ).

Proof of Lemma 6.4. We will first prove that $\theta \leq \frac{3}{5}n'$. Suppose not. Lemma 6.1(iii) says that the sum of $\overline{A}$-degrees of the non-huge degree vertices in A_2 is $\rho \leq n' - \theta < \theta$. Since, by construction, $e(A_1, \overline{A}) \geq e(A_2, \overline{A})$, it is impossible for both of the huge $\overline{A}$-degree vertices to be in A_1, because it implies that

$$\theta = e(A_1, \overline{A}) - e(A_2, \overline{A}) \geq 2\theta - \rho > \theta\,.$$

which is a contradiction. Thus, the only remaining case is when A_1 and A_2 each have a huge-degree vertex. Then since $e(A_1, \overline{A}) \leq n'$ and $e(A_2, \overline{A}) \geq \theta$, the maximum possible gap is

$$\theta = e(A_1, \overline{A}) - e(A_2, \overline{A}) \leq n' - \theta < \theta\,,$$

which again is a contradiction.

Now, using $\theta \leq \frac{3}{5}n'$ and Lemma 6.3 with $\alpha = 2$, $\delta \geq 4$, we will verify (12) which says,

$$\theta + \frac{\tau}{2} \leq \frac{n}{2} + \frac{m}{\delta+1}.$$

Note that by Lemmas 6.1 and 6.3, we have $\tau \leq \frac{n'+\rho}{\delta-\alpha+1} = \frac{n'+\rho}{\delta-1} \leq \frac{2n'-\theta}{\delta-1} \leq \frac{2n'-\theta}{3}$. By the minimum degree assumption, we also have $m \geq \frac{\delta}{2}n'$. Thus it suffices to prove that

$$\theta + \frac{2n'-\theta}{6} \leq \frac{n'}{2} + \frac{\delta n'}{2(\delta+1)},$$

and since $\delta \geq 4$, that $\frac{5}{6}\theta \leq \left(\frac{1}{2} + \frac{2}{5} - \frac{1}{3}\right)n' = \frac{17}{30}n'$. This is true as $\theta \leq \frac{3}{5}n'$. □

6.5 Convexity extremes

In this section, we prove Lemma 6.5, which disposes of the $\alpha = 3$ and $\alpha = \delta - 1$ cases of (17) when $\delta \geq 6$. We begin with $\alpha = 3$, which corresponds to

Lemma 6.8. *Let $\delta \geq 5$, and let n', θ, ρ be non-negative real numbers satisfying $\theta \leq n'$ and $\rho \leq n' - \theta$. Then*

$$\theta + \frac{n' + \rho}{2(\delta - 2)} \leq \frac{2\delta + 1}{2(\delta + 1)} n' + \frac{3\theta + \rho}{2(\delta + 1)} .$$

Proof. The coefficient of ρ on the left hand side is clearly larger than its coefficient on the right hand side, so the inequality is sharpest when ρ hits its constraint of $n' - \theta$. It then suffices to show

$$\theta + \frac{2n' - \theta}{2(\delta - 2)} \leq \frac{2\delta + 1}{2(\delta + 1)} n' + \frac{2\theta + n'}{2(\delta + 1)} .$$

Next, observe that the coefficient of θ on the left hand side is $\frac{2\delta - 5}{2(\delta - 2)}$, while its coefficient on the right hand side is $\frac{2}{2(\delta + 1)}$. The left-hand coefficient is easily larger in our range $\delta \geq 5$, so the inequality sharpens when we increase θ to its constraint of n'. It therefore remains to show that

$$n' + \frac{n'}{2(\delta - 2)} \leq \frac{2\delta + 1}{2(\delta + 1)} n' + \frac{3n'}{2(\delta + 1)} ,$$

or equivalently,

$$\frac{2\delta - 3}{2(\delta - 2)} \leq \frac{2\delta + 4}{2(\delta + 1)} ,$$

which one easily verifies is true for all $\delta \geq 5$. □

Next we establish the $\alpha = \delta - 1$ case.

Lemma 6.9. *Let $\delta \geq 5$, and let n', θ, ρ be non-negative real numbers satisfying $\theta \leq n'$ and $\rho \leq n' - \theta$. Then*

$$\theta + \frac{n' + \rho}{4} \leq \frac{2\delta + 1}{2(\delta + 1)} n' + \frac{(\delta - 1)\theta + \rho}{2(\delta + 1)} .$$

Proof. The coefficient of ρ on the left hand side is clearly greater than its coefficient on the right hand side, so we may again assume that $\rho = n' - \theta$. It remains to show

$$\theta + \frac{2n' - \theta}{4} \leq \frac{2\delta + 1}{2(\delta + 1)} n' + \frac{(\delta - 2)\theta + n'}{2(\delta + 1)} .$$

The coefficient of θ on the left hand side is $\frac{3}{4}$, which is again easily greater than its coefficient of $\frac{\delta - 2}{2(\delta + 1)}$ on the right hand side; hence we may assume $\theta = n'$, leaving us to establish

$$n' + \frac{n'}{4} \leq \frac{2\delta + 1}{2(\delta + 1)} n' + \frac{(\delta - 1)n'}{2(\delta + 1)} ,$$

which is equivalent to

$$\frac{5}{4} \leq \frac{3\delta}{2(\delta + 1)} .$$

One easily sees that this indeed holds for $\delta \geq 5$. □

6.6 Tight components with three huge degree vertices

We prove Lemma 6.6 for the case $\delta = 4$ and $\alpha = 3$. Using these parameters in (19) and (16), we see that we need to establish

$$\theta + \frac{1}{2}\left(\frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots\right) \le \frac{n'}{2} + \frac{1}{5}\left(3\theta + \rho + \frac{1}{2}(d_1 + 2d_2 + 3d_3 + \cdots)\right), \tag{22}$$

subject to the constraints

$$d_0 \le \rho, \qquad \theta \le n', \qquad \rho \le n' - \theta, \quad \text{and} \quad \sum_i d_i = n'. \tag{23}$$

We consider the differences between the coefficients of the d_i on the left and on the right. For d_0, its coefficient on the left is $\frac{1}{2}$, while its coefficient on the right is 0. This is the only coefficient where that differential is in favor of the left hand side. Indeed, it is clear that for even i, the difference between the left hand coefficient and the right hand coefficient strictly decreases as i increases. A similar fact is true for odd i. Note that the differential for the coefficient of d_1 is $0 - \frac{1}{10}$, whereas the differential for d_2 is $\frac{1}{6} - \frac{1}{5} = -\frac{1}{30}$. Hence the differential is largest for d_2 (apart from d_0) since $-\frac{1}{30} > -\frac{1}{10}$.

Therefore, in view of the constraints $d_0 \le \rho$ and $\sum_i d_i = n'$, inequality (22) is sharpest when d_0 is as large as possible, and all the rest of the weight is on d_2. It remains to show that

$$\theta + \frac{1}{2}\left(\rho + \frac{n' - \rho}{3}\right) \le \frac{n'}{2} + \frac{1}{5}\left(3\theta + \rho + (n' - \rho)\right),$$

or equivalently,

$$\theta + \frac{\rho}{3} + \frac{n'}{6} \le \frac{7}{10}n' + \frac{3}{5}\theta.$$

Since ρ only appears on the left hand side, the inequality is sharpest when ρ hits its upper constraint of $n' - \theta$; it remains to show

$$\frac{2}{3}\theta + \frac{n'}{2} \le \frac{7}{10}n' + \frac{3}{5}\theta.$$

This is equivalent to $\frac{1}{15}\theta \le \frac{1}{5}n'$, which clearly holds since $\theta \le n'$. □

6.7 Tight components with a single vertex of huge degree

We now prove our sharpest estimate (Lemma 6.6), which handles the situation when A contains a single vertex of huge degree. As we need to establish (19), we rearrange it to isolate the involvement of the degree sequence. We must show:

$$\frac{1}{2}\left(\frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots\right) - \frac{1}{2(\delta+1)}(d_1 + 2d_2 + 3d_3 + \cdots) \le \frac{n'}{2} + \frac{\theta + \rho}{\delta + 1} - \theta. \tag{24}$$

We first control the left hand side in terms of ρ, using the given constraint (16).

Lemma 6.10. *For every positive even integer δ and non-negative real numbers $n', \rho, d_0, d_1, \ldots$, the maximum value of the following objective:*

$$\frac{1}{2}\left(\frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots\right) - \frac{1}{2(\delta+1)}(d_1 + 2d_2 + 3d_3 + \cdots),$$

subject to the constraints

$$\sum_{i=0}^{\delta-2} (\delta - 1 - i) d_i \leq \rho\,, \quad \textit{and} \quad \sum_i d_i = n'\,,$$

is at most

$$-\frac{\delta-1}{2(\delta+1)} n' + \frac{\delta}{(\delta-1)(\delta+1)} \rho\,. \tag{25}$$

Proof. Let us study the coefficients of the d_i in the objective. Coincidentally, the coefficients of $d_{\delta-1}$ and d_δ are both equal to

$$0 - \frac{1}{2(\delta+1)}(\delta-1) = -\frac{\delta-1}{2(\delta+1)} = \frac{1}{2}\cdot\frac{1}{\delta+1} - \frac{1}{2(\delta+1)}\cdot\delta\,,$$

while all coefficients of d_i with $i \geq \delta+1$ are strictly smaller. Therefore, by moving weight from d_i to $d_{\delta-1}$ so that $\sum_i d_i$ remains constant, we can increase the objective function without violating any constraints. This allows us to conclude that there is an optimal point which has $d_i = 0$ for all $i \geq \delta$.

We aim to show that there is in fact an optimizer which is supported only at d_0 and $d_{\delta-1}$. So, suppose that the optimizer has $d_i > 0$ for some $1 \leq i \leq \delta - 2$. Consider reducing d_i to 0, increasing d_0 by $\frac{\delta-1-i}{\delta-1} d_i$, and increasing $d_{\delta-1}$ by $\frac{i}{\delta-1} d_i$. (This preserves both $\sum_{i=0}^{\delta-2}(\delta-1-i)d_i$ and $\sum_i d_i$.) Note that the coefficients of d_0 and $d_{\delta-1}$ in the objective are $\frac{1}{2}$ and $-\frac{\delta-1}{2(\delta+1)}$, respectively.

Case 1: *i* is odd. The coefficient of d_i in the objective is $-\frac{i}{2(\delta+1)}$, so this perturbation increases the objective by:

$$\begin{aligned} & d_i \left[\frac{1}{2}\cdot\frac{\delta-1-i}{\delta-1} - \frac{i}{2(\delta+1)}\cdot(-1) + \left(-\frac{\delta-1}{2(\delta+1)}\right)\cdot\frac{i}{\delta-1}\right] \\ &\quad = d_i \left[\frac{(\delta-1-i)(\delta+1) + i(\delta-1) - (\delta-1)i}{2(\delta-1)(\delta+1)}\right] \\ &\quad = d_i \cdot \frac{\delta-1-i}{2(\delta-1)}\,, \end{aligned}$$

which is clearly positive for $i \leq \delta - 2$.

Case 2: *i* is even. The coefficient of d_i in the objective is now $\frac{1}{2(i+1)} - \frac{i}{2(\delta+1)}$, so by the previous calculation this perturbation increases the objective by:

$$d_i \cdot \frac{\delta-1-i}{2(\delta-1)} - d_i \cdot \frac{1}{2(i+1)}\,,$$

which is non-negative precisely when

$$\frac{\delta-1-i}{2(\delta-1)} \geq \frac{1}{2(i+1)}\,. \tag{26}$$

Yet it is clear that (26) has equality at both $i = 0$ and $i = \delta - 2$. Since the left hand side is linear in i, while the right hand side is convex on the domain $i > -1$, we immediately conclude that (26) holds for all $0 \leq i \leq \delta - 2$.

As both the odd and even cases produce perturbations that do not decrease the objective, we may therefore consider an optimizer which is entirely supported at d_0 and $d_{\delta-1}$. The constraints then translate into

$$(\delta-1)d_0 \le \rho \quad \text{and} \quad d_0 + d_{\delta-1} = n' .$$

Since the coefficient of d_0 in the objective $\left(\frac{1}{2}\right)$ is strictly larger than that of $d_{\delta-1}$ (which is $-\frac{\delta-1}{2(\delta+1)}$), to maximize the objective function, we need to take d_0 to be as large as possible, i.e., $d_0 = \frac{\rho}{\delta-1}$, and then choose $d_{\delta-1}$ to satisfy $d_0 + d_{\delta-1} = n'$. Hence we conclude that the objective is indeed bounded from above by

$$\frac{1}{2} \cdot \frac{\rho}{\delta-1} - \frac{\delta-1}{2(\delta+1)} \cdot \left(n' - \frac{\rho}{\delta-1}\right) ,$$

which is precisely (25). □

Our final ingredient for the proof of Theorem 1.9 now follows easily.

Proof of Lemma 6.6. Section 6.6 establishes the result for the case $(\delta, \alpha) = (4, 3)$, so it remains to consider the case when $\alpha = 1$. Using the result of Lemma 6.10 in (24), we see that it remains to establish

$$-\frac{\delta-1}{2(\delta+1)}n' + \frac{\delta}{(\delta-1)(\delta+1)}\rho \le \frac{n'}{2} + \frac{\theta+\rho}{\delta+1} - \theta ,$$

over the domain bounded by $0 \le \theta \le n'$ and $\rho \le n' - \theta$. Equivalently, we need

$$\frac{\delta}{\delta+1}\theta + \frac{1}{(\delta-1)(\delta+1)}\rho \le \frac{\delta}{\delta+1}n' .$$

Using $\rho \le n' - \theta$ to eliminate ρ, we see that it suffices to prove

$$\left(\frac{\delta}{\delta+1} - \frac{1}{(\delta-1)(\delta+1)}\right)\theta \le \left(\frac{\delta}{\delta+1} - \frac{1}{(\delta-1)(\delta+1)}\right)n' ,$$

which clearly follows from $\theta \le n'$. This completes our proof. □

7 Analysis of extremal examples

In this section, we establish the asymptotic extremality of the two families of constructions which we described after Theorem 1.9 in the Introduction.

Proof of Proposition 1.10. Let $\delta \ge 2$ be a fixed even number. We begin with the first family. Let x and y be non-negative integers, with y odd. Let $G = (V, E)$ be the vertex-disjoint union of x copies of K_δ and y copies of $K_{\delta+1}$, together with a new vertex v_0 which is adjacent to all other vertices. The number of vertices is then

$$n = \delta x + (\delta+1)y + 1 ,$$

which is even, and the number of edges is

$$m = \binom{\delta}{2}x + \binom{\delta+1}{2}y + \delta x + (\delta+1)y = \frac{\delta(\delta+1)}{2}x + \frac{(\delta+1)(\delta+2)}{2}y .$$

Let $V = V_1 \cup V_2$ be an arbitrary bisection, and without loss of generality, suppose that the dominating vertex v_0 lies in V_1. The induced subgraph $G[V_1 \setminus \{v_0\}]$ is then the union of $t \le x + y$ cliques, with orders $c_1, c_2, \ldots, c_t$. Since we have a bisection, $\sum c_i = \frac{n}{2} - 1$. Using convexity, it is then clear that $\sum \binom{c_i}{2}$ is minimized when $t = x+y$, and the c_i are within 1 of each other. Since $\frac{n}{2} - 1 = \frac{\delta}{2}(x+y) + \frac{y-1}{2}$, this happens precisely when $x + \frac{y+1}{2}$ of the c_i's are $\frac{\delta}{2}$, and $\frac{y-1}{2}$ of the c_i's are $\frac{\delta}{2} + 1$. Therefore,

$$\begin{aligned}
e(V_1) &= \left(\frac{n}{2} - 1\right) + \sum_{i=1}^{t} \binom{c_i}{2} \\
&\ge \left(\frac{\delta x + (\delta+1)y + 1}{2} - 1\right) + \left(x + \frac{y+1}{2}\right)\binom{\delta/2}{2} + \frac{y-1}{2}\binom{(\delta/2)+1}{2} \\
&= x\left[\frac{\delta}{2} + \binom{\delta/2}{2}\right] + y\left[\frac{\delta+1}{2} + \frac{1}{2}\binom{\delta/2}{2} + \frac{1}{2}\binom{(\delta/2)+1}{2}\right] \\
&\quad + \left[-\frac{1}{2} + \frac{1}{2}\binom{\delta/2}{2} - \frac{1}{2}\binom{(\delta/2)+1}{2}\right] \\
&= x \cdot \frac{\delta(\delta+2)}{8} + y \cdot \frac{(\delta+2)^2}{8} - \frac{\delta+2}{4} \\
&= \frac{\delta+2}{4(\delta+1)} m - \frac{\delta+2}{4},
\end{aligned}$$

which is indeed $\left(\frac{\delta+2}{4(\delta+1)} - o(1)\right)m$ since δ is constant.

Proceeding to the second family, let $G = (V, E)$ be the complete bipartite graph $K_{\delta+1,n-\delta-1}$. The number of edges is

$$m = (\delta+1)(n-\delta-1).$$

Consider a bisection $V = V_1 \cup V_2$. Of the $\delta + 1$ vertices from the smaller part of the $K_{\delta+1,n-\delta-1}$, suppose that x of them are in V_1 and y of them are in V_2. Without loss of generality, suppose that $x \ge y$; then, $x \ge \frac{\delta}{2} + 1$, and

$$e(V_1) = x\left(\frac{n}{2} - x\right) \ge \left(\frac{\delta}{2} + 1\right)\left(\frac{n}{2} - \frac{\delta}{2} - 1\right) = \frac{\delta+2}{4(\delta+1)} m - \frac{\delta+2}{4}.$$

Note that the bound we obtained from the two families are exactly the same.

8 Related bisection results

In this section, we prove the results which were stated in Section 1.3. We begin by proving that $K_{n/3,2n/3}$ has a bisection of size at least $\frac{3}{4}m$, but every bisection induces at least $\frac{m}{4}$ edges on some side.

Proof of Proposition 1.12. In the graph $K_{n/3,2n/3}$, let A be the part of size $\frac{n}{3}$ and B be the part of size $\frac{2n}{3}$. The total number of edges is $m = \frac{2n^2}{9}$. The largest bisection is obtained by placing all the vertices of A on one side and arbitrarily allocating the vertices of B to create a bisection. This produces a bisection of size $\frac{n}{3} \cdot \frac{n}{2} = \frac{n^2}{6} = \frac{3}{4}m$.

Now consider a bisection $V_1 \cup V_2$. Without loss of generality we may assume that V_1 contains at least $xn \ge \frac{|A|}{2} = \frac{n}{6}$ vertices of A (thus $\frac{1}{6} \le x \le \frac{1}{3}$). Then the number of edges in V_1 is

$$e(V_1) = xn \cdot \left(\frac{n}{2} - xn\right) = x\left(\frac{1}{2} - x\right)n^2 \ge \frac{1}{18}n^2 = \frac{m}{4}.$$

Therefore, $\max\{e(V_1), e(V_2)\} \geq \frac{m}{4}$ for every bisection of the vertex set. □

Next, we prove Theorem 1.13, which finds almost-bisections of size at least $\frac{r+1}{2r}m$ or $\frac{r+2}{2(r+1)}m$ (depending on the parity of r) in graphs with degrees bounded by r.

Proof of Theorem 1.13. Let $G = (V, E)$ be a graph of maximum degree at most r. The Hajnal-Szemerédi theorem [15], produces an *equitable* coloring of G with $r+1$ colors, i.e., a coloring where all color class sizes differ by at most one. Let $W_1, W_2, \ldots, W_{r+1}$ be the color classes.

Consider first the case when r is odd. Generate a random bipartition $V = V_1 \cup V_2$ by uniformly selecting exactly $\frac{r+1}{2}$ of the color classes to put in V_1, and placing the other $\frac{r+1}{2}$ color classes in V_2. The probability that a fixed edge contributes towards $e(V_1, V_2)$ is exactly $(\frac{r+1}{2})^2/\binom{r+1}{2} = \frac{r+1}{2r}$, so by linearity of expectation, the expected value of $e(V_{1,}V_2)$ is $\frac{r+1}{2r}m$. Consequently, there exists a bisection which contains at least so many edges. Since all color classes differ in size by at most 1, we automatically have that the V_i differ in size by at most $\frac{r+1}{2}$, as claimed.

Now consider the case when r is even. Generate a random bisection $V_1 \cup V_2$ of G as follows. First, pick a special index $k \in [r+1]$ uniformly at random, generate a random bisection of W_k, and put the parts into V_1 and V_2. Then, uniformly distribute the remaining color classes as in the odd case above. There are two ways for a fixed edge e to contribute to $e(V_1, V_2)$. The probability that one endpoint lies in W_k is $\frac{2}{r+1}$, and then the other endpoint has probability $\frac{1}{2}$ of being distributed onto the other side of the bipartition. On the other hand, the probability that neither endpoint lies in W_k, and the endpoint color classes are separated across the bipartition, is exactly $\frac{r-1}{r+1} \cdot \frac{r^2/4}{\binom{r}{2}} = \frac{r}{2(r+1)}$. Therefore, the probability that e contributes to $e(V_1, V_2)$ is exactly

$$\frac{2}{r+1} \cdot \frac{1}{2} + \frac{r}{2(r+1)} = \frac{r+2}{2(r+1)}\,.$$

Consequently, the expected value of $e(V_1, V_2)$ is $\frac{r+2}{2(r+1)}m$, and there exists a bisection which contains at least so many edges. Again, the gap between the $|V_i|$ is at most $1 + \frac{r}{2}$, because the bisection of W_k can introduce an error of at most 1, and the other r color classes differ in size by at most 1. □

The previous bound easily produces a result on perfect bisections of bounded degree graphs.

Corollary 8.1. *If G has maximum degree at most r, then there exists a bisection of size at least $\frac{r+1}{2r}m - \frac{r(r+1)}{4}$ if r is odd and $\frac{r+2}{2(r+1)}m - \frac{r(r+2)}{4}$ if r is even.*

Proof. If r is odd, the previous theorem produces a bipartition of size at least $\frac{r+1}{2r}m$, and sides differing in size by at most $\frac{r+1}{2}$. We may then balance the bipartition by moving at most $\frac{r+1}{4}$ vertices. Since all degrees are bounded by r, this affects the size of the cut by at most $\frac{r(r+1)}{4}$, as claimed.

On the other hand, if r is even, then the previous theorem produces a bipartition of size at least $\frac{r+2}{2(r+1)}m$, which can be balanced by moving at most $\frac{r+2}{4}$ vertices. By a similar argument to above, this affects the cut size by at most $\frac{r(r+2)}{4}$, as claimed. □

Finally, we prove that every r-regular graph has a bisection of size at least $\frac{r+1}{2r}m$ when r is odd and $\frac{r+2}{2(r+1)}m$ when r is even.

Proof of Theorem 1.14. An r-regular graph clearly has chromatic number $k \leq r+1$. Thus by the relation between Max Cut and the chromatic number of a graph mentioned in the introduction (see the discussion following Proposition 1.12, or [2, 20, 21]), we can find a bipartition $V = V_1 \cup V_2$ with the required size, but where the part sizes are not necessarily the same. It suffices to prove that we can always move vertices from the larger part (V_1, say) to the smaller part (V_2, say) while keeping the property that the bipartition has at least the required size. Indeed, if there is a vertex $v \in V_1$ with at least as many neighbors in V_1 as in V_2, then moving v to V_2 would not decrease the size of the bipartition. Repeatedly move such vertices until either we reach a bisection and are done, or there are no more such vertices.

In the later case, every vertex in V_1 has more neighbors in V_2 than in V_1. Move $|V_1| - \frac{n}{2}$ vertices of V_1 to V_2. Note that the number of neighbors in V_2 of any of the remaining $\frac{n}{2}$ vertices of V_1 only grows. Therefore, each of them still have strictly more than $\frac{r}{2}$ neighbors in V_2. Yet G was r-regular, so $m = \frac{rn}{2}$. If r is odd, the number of crossing edges is then at least $\frac{n}{2} \cdot \frac{r+1}{2} = \frac{r+1}{2r}m$. If r is even, the number of crossing edges is at least $\frac{n}{2} \cdot \left(\frac{r}{2} + 1\right) = \frac{r+2}{2r}m > \frac{r+2}{2(r+1)}m$. Therefore, in both cases we obtain a bisection, while still maintaining the desired cut size. □

9 Concluding remarks

In this paper, we studied the graph bisection problem from several different angles. Theorem 1.3 extended the classical Edwards bound for graph bipartition by introducing the maximum degree, together with a new parameter, the number of tight components. These tight components were very useful in our analysis, but have not been widely studied in the literature. Indeed, we are not aware of results concerning their general structure. For example, the simplest example of a tight component is a clique of odd order, and more complex tight components can be assembled by taking two vertex-disjoint tight components, and identifying one vertex from each of them. Yet it is not clear if every tight component can be obtained through such a process.

Using our parameterization in terms of the number of tight components, we then pushed our approach further to study judicious bisections of graphs with respect to minimum-degree constraints (Theorem 1.9). Part of the difficulty in proving this result stemmed from the wide range of extremal examples, as described in Proposition 1.10. From our proof, it appears that these examples are in some sense exhaustive. More precisely, when the inequalities are tight (up to the $o(m)$ error terms), we either have (i) exactly $\delta + 1$ large degree vertices, each of degree about n, with no other edges, or (ii) exactly one vertex of large degree, with degree about n, and with the correct number of tight components in the remainder. However, since the error term still remains in Theorem 1.9, we leave this as only an observation, and ask whether one can remove the error term from Theorem 1.9. More precisely, is it true that for every positive even integer δ, every graph of minimum degree at least δ admits a bisection in which the number of edges within each part is at most $\frac{\delta+2}{4(\delta+1)}m$?

For directed graphs, one can also ask questions in which one seeks to maximize or minimize several quantities simultaneously. Indeed, consider the following problem (from, e.g., [25]), which asks to partition the vertex set of a digraph D into two parts V_1 and V_2 such that both $e(V_1, V_2)$ and $e(V_2, V_1)$ are large. It appears that some of the ideas developed in this paper can be used to attack this problem. We will pursue this approach further in a subsequent paper.

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