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Large induced trees in K_r -free graphs

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ABSTRACT

For a graph G , let $t(G)$ denote the maximum number of vertices in an induced subgraph of G that is a tree. In this paper, we study the problem of bounding $t(G)$ for graphs which do not contain a complete graph K_r on r vertices. This problem was posed twenty years ago by Erdős, Saks, and Sós. Substantially improving earlier results of various researchers, we prove that every connected triangle-free graph on n vertices contains an induced tree of order \sqrt{n} . When $r \geq 4$, we also show that $t(G) \geq \frac{\log n}{4 \log r}$ for every connected K_r -free graph G of order n . Both of these bounds are tight up to small multiplicative constants, and the first one disproves a recent conjecture of Matoušek and Šámal.

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1. Introduction

For a graph G , let $t(G)$ denote the maximum number of vertices in an induced subgraph of G that is a tree. The problem of bounding $t(G)$ in a connected graph G was first introduced twenty years ago by Erdős, Saks, and Sós [5]. Clearly, to get a non-trivial result one must impose some conditions on the graph G , because, for example, the complete graph contains no induced tree with more than 2 vertices. In their paper, Erdős, Saks, and Sós studied the relationship between $t(G)$ and several natural parameters of the graph G . They were able to obtain asymptotically tight bounds on $t(G)$ when either the number of edges or the independence number of G were known.

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Erdős, Saks, and Sós also considered the problem of estimating the size of the largest induced tree in graphs with no K_r (complete graph on r vertices). Let $t_r(n)$ be the minimum value of $t(G)$ over all connected K_r -free graphs G on n vertices. In particular, for triangle-free graphs, they proved that

$$\Omega\left(\frac{\log n}{\log \log n}\right) \leq t_3(n) \leq O(\sqrt{n} \log n),$$

and left as an interesting open problem the task of closing the wide gap between these two bounds.

The first significant progress on this question was made only recently by Matoušek and Šámal [15], who actually came to the problem of estimating $t_3(n)$ from a different direction. Pultr had been studying forbidden configurations in Priestley spaces [2], and this led him to ask in [17] how large $t(G)$ could be for connected bipartite graphs G . Let $t_B(n)$ be the minimum value of $t(G)$ over all connected bipartite graphs on n vertices. It is clear that $t_3(n) \leq t_B(n)$, so the result of Erdős, Saks, and Sós immediately gives a lower bound on $t_B(n)$.

Motivated by Pultr’s question, Matoušek and Šámal studied $t_B(n)$ and $t_3(n)$. They found the following nice construction which shows that $t_3(n) \leq t_B(n) < 2\sqrt{n} + 1$. Let $m = \sqrt{n}$, and consider the graph with parts $V_{-m+1}, V_{-m+2}, \dots, V_{m-1}$, where $|V_i| = m - |i|$, and each consecutive pair of parts (V_i, V_{i+1}) induces a complete bipartite graph. This graph is clearly bipartite with $m^2 = n$ vertices, and it is easy to see that every induced tree in it has at most $2m - 1$ vertices. On the other hand, Matoušek and Šámal were able to improve the lower bound on $t_B(n)$ and $t_3(n)$, showing that $t_3(n) \geq e^{c\sqrt{\log n}}$ for some constant c . Furthermore, they also proved that if there was even a single value of n_0 for which $t_3(n_0) < \sqrt{n_0}$, then in fact $t_3(n) \leq O(n^\beta)$ for some constant β strictly below $1/2$. The above fact led Matoušek and Šámal to conjecture that the true asymptotic behavior of $t_3(n)$ was some positive power of n which is strictly smaller than $1/2$.

Our first main result essentially solves this problem. It determines that the order of growth of both $t_3(n)$ and $t_B(n)$ is precisely $\Theta(\sqrt{n})$, disproving the conjecture of Matoušek and Šámal.

Theorem 1. *Let G be a connected triangle-free graph on n vertices. Then $t(G) \geq \sqrt{n}$.*

Furthermore, our approach can also be used to give asymptotically tight bounds on the size of the largest induced tree in K_r -free graphs for all remaining values of r . In their original paper, Erdős, Saks, and Sós gave an elegant construction which shows that $t_r(n)$ for $r \geq 4$ has only logarithmic growth. Indeed, let T be a balanced $(r - 1)$ -regular tree, that is, a rooted tree in which all non-leaf vertices have degree $r - 1$ and the depth of any two leaves differs by at most 1. Then the line graph⁴ $L(T)$ is clearly K_r -free, and one can easily check that induced trees in $L(T)$ correspond to induced paths in T , which have only logarithmic length. Optimizing the choice of the parameters in this construction, one can show that $t_r(n) \leq \frac{2 \log(n-1)}{\log(r-2)} + 2$. On the other hand, using Ramsey Theory, Erdős, Saks, and Sós also showed that $t_r(n) \geq \frac{c_r \log n}{\log \log n}$, where c_r is a constant factor depending only on r . Our second main result closes the gap between these two bounds as well, and determines the order of growth of $t_r(n)$ up to a small multiplicative constant.

Theorem 2. *Let $r \geq 4$, and let G be a connected graph on n vertices with no clique of size r . Then $t(G) \geq \frac{\log n}{4 \log r}$.*

One can also study induced forests rather than trees in K_r -free graphs. Let $f_r(n)$ be the maximum number such that every K_r -free graph on n vertices contains an induced forest with at least $f_r(n)$ vertices. Trivially we have $f_r(n) \geq t_r(n)$, but it appears that the size of the maximum induced forest in a graph is more closely related to another parameter. The *independence number* $\alpha(G)$ of a graph is the size of the largest independent set of vertices in G . Since an independent set is a forest and every forest is bipartite, the size of the largest induced forest in a graph G is at least $\alpha(G)$ and at most $2\alpha(G)$. Using the best known upper bound for off-diagonal Ramsey numbers [1], for fixed $r \geq 3$

⁴ The vertices of $L(T)$ are the edges of T , and two of them are adjacent if they share a vertex in T .

and all n we have $f_r(n) \geq cn^{\frac{1}{r-1}} \log^{\frac{r-2}{r-1}} n$ for some positive constant c . Hence, $f_3(n)$ is larger than $t_3(n)$ by a factor of $c\sqrt{\log n}$. Furthermore, for fixed $r > 3$, $f_r(n)$ and $t_r(n)$ behave very differently, as $f_r(n)$ is polynomial in n while $t_r(n)$ is only logarithmic. This demonstrates that in K_r -free graphs the largest guaranteed induced forest is much larger than the largest guaranteed induced tree.

We close this introduction by mentioning some related research. Our work considers the Ramsey-type problem of finding either a clique or a large induced tree. The similar problem of finding an induced copy of a *particular* tree T in a K_r -free graph was independently raised by Gyárfás [9] and Sumner [20]. They conjectured that for any fixed integer r and tree T , any graph with sufficiently large chromatic number (depending on r and T) must contain either an r -clique or an induced copy of T . Note that the essential parameter for the graph G is now the chromatic number and not the number of vertices. Indeed, a complete bipartite graph has no clique of size 3, but contains only stars as induced subtrees. This conjecture is widely open, although some partial results were obtained in [10–12,18].

Induced trees were also studied in the context of sparse random graphs. This line of research was started by Erdős and Palka [4], who conjectured that for any constant $c > 1$, the random graph $G(n, c/n)$ would with high probability contain an induced tree of order $\gamma(c)n$. This was solved by Fernandez de la Vega [6], and other variants of this result were obtained in [7,8,13,14,19]. In another regime, when the edge probability is $p = c \log n/n$, Palka and Ruciński [16] showed that the largest induced tree in $G(n, p)$ has size $\Theta(n \log \log n / \log n)$ with high probability.

The rest of this paper is organized as follows. In Section 2 we discuss the proof of Theorem 1, and show how to reduce it to an abstract optimization problem on certain bipartite graphs with weights on the vertices. The solution of this problem is provided in the following section. In Section 4, we show how to extend our argument to the case of K_r -free graphs with $r \geq 4$, and prove our second result, Theorem 2. The final section of the paper contains a few concluding remarks. Throughout our paper, we will omit floor and ceiling signs whenever they are not essential, to improve clarity of presentation.

2. Triangle-free graphs

The main idea in the proof of Theorem 1 is to use induction to prove a slightly⁵ stronger statement. Instead of finding a single induced tree, we show that no matter which vertex v of the graph we choose, there exists a large induced tree which contains v . More precisely, we prove that any connected, triangle-free graph with $n + 1$ vertices contains an induced tree of size $\sqrt{n} + 1$ through any given vertex.

This is obviously true for $n = 1$, which serves as the base of our induction. It remains to prove the statement for general $n \geq 2$, while assuming its truth for all smaller values of n . So, let $G = (V, E)$ be an arbitrary connected triangle-free graph with $n + 1$ vertices, and fix an arbitrary vertex $v \in V$. We will find a large induced tree through v . Note that since G is triangle-free, $\{v\} \cup N(v)$ induces a star. Therefore, we may assume that the size of the neighborhood satisfies $|N(v)| < \sqrt{n}$, or else we are done.

Consider the subgraph of G induced by $V \setminus (\{v\} \cup N(v))$. It decomposes into connected components, whose vertex sets we call V_1, \dots, V_m . Now suppose that we could find a subset $U \subset N(v)$, and a subset $I \subset [m]$, with the following properties:

- (i) For each $i \in I$, there is exactly one $u \in U$ which is adjacent to at least one vertex in V_i . Let us denote this vertex by $u(i)$.
- (ii) The sum $\sum_{i \in I} \sqrt{|V_i|}$ is at least $\sqrt{|V_1 \cup \dots \cup V_m|}$.

Then, for each $i \in I$, we could apply the induction hypothesis to the connected subgraph of G induced by $\{u(i)\} \cup V_i$. This would give an induced tree T_i containing $u(i)$, of size $1 + \sqrt{|V_i|}$. Furthermore, it is easy to see that the union of $\{v\}$ with all of the above constructed trees T_i is also an induced tree.

⁵ We will discuss the relative strength of this statement in detail in our concluding remarks.

Indeed, since each V_i is a maximal connected component, there are no edges between the T_i , and since G is triangle-free, there are no edges inside $U \subset N(v)$. Therefore, we will have an induced tree with total size at least:

$$\begin{aligned} |\{v\}| + |\{u(i): i \in I\}| + \sum_{i \in I} \sqrt{|V_i|} &\geq 1 + 1 + \sqrt{|V_1 \cup \dots \cup V_m|} \\ &= 2 + \sqrt{|V \setminus (\{v\} \cup N(v))|} \\ &\geq 2 + \sqrt{(n+1) - 1 - \sqrt{n}} \\ &\geq 1 + \sqrt{n}, \end{aligned}$$

as desired. Thus, the following abstract lemma completes the proof.

Lemma 1. Consider a bipartite graph with sides A and B , with the property that each vertex in B has degree at least 1. Let each vertex $i \in B$ have an associated weight $w_i \geq 0$. We call a subset $H \subset A \cup B$ **admissible** if each vertex $v \in B \cap H$ has exactly one neighbor in $A \cap H$. Then there exists an admissible H with $\sum_{i \in B \cap H} \sqrt{w_i} \geq \sqrt{\sum_{i \in B} w_i}$.

The connection between this lemma and our required selection of $I \subset [m]$ and $U \subset N(v)$ is clear. The sides A and B correspond to the sets $N(v)$ and $[m]$, respectively, and the weights w_i are precisely the sizes of the connected components $|V_i|$. The requirement that each vertex in B has degree at least 1 is satisfied by the fact that G is connected, and so each component V_i has at least one neighbor in $N(v)$. Therefore, this lemma will indeed complete the proof of Theorem 1.

3. Main lemma

Before proving our main lemma, Lemma 1, let us discuss an easy special case which we will actually need later in our study of K_r -free graphs when $r \geq 4$. Observe that if the weights w_i in Lemma 1 were roughly equal, then one way to control the objective $\sum_{i \in B \cap H} \sqrt{w_i}$ would be to find a lower bound on $|B \cap H|$. This motivates the following claim, which we record for later use.

Lemma 2. Consider a bipartite graph with sides A and B , with the property that each vertex in B has degree at least 1. We still call a subset $H \subset A \cup B$ **admissible** if each vertex $v \in B \cap H$ has exactly one neighbor in $A \cap H$. Then there exists an admissible H with $|B \cap H| \geq \sqrt{|B|}$.

Proof. The key observation, which we will also use in the proof of Lemma 1, is that we may assume that every $v \in A$ has some vertex $w \in B$ which is adjacent only to v . Indeed, suppose this is not the case, and every neighbor of v has additional neighbors in A . Then, deleting v from A will not break the hypothesis of the lemma. Therefore, after repeatedly performing this reduction, we obtain a bipartite graph in which every vertex $v \in A$ has a neighbor in B that sees only v . Notice that this implies that there is an induced matching between A and some subset $B' \subset B$.

If $|A| \geq \sqrt{|B|}$, then the induced matching immediately yields an admissible set $H = A \cup B'$ which satisfies the assertion. On the other hand, when $|A| < \sqrt{|B|}$, there is a vertex in A with degree at least $\sqrt{|B|}$. Indeed, since every vertex in B has degree at least 1, the total number of edges in the bipartite graph is at least $|B|$, and therefore some vertex $v \in A$ has degree $\geq |B|/|A| > \sqrt{|B|}$. The induced star $H = \{v\} \cup N(v)$ provides the desired admissible set. \square

We pause now to remark that Lemma 2 is far from being sharp. In fact, it is always possible to find an admissible H with $|H \cap B| \geq \Omega(|B|/\log |B|)$, and this is tight. Although we do not need this result for our proof we sketch it here for the sake of completeness. By the reduction in the proof of Lemma 2, we may assume that there is an induced matching between A and a subset $B' \subset B$. In particular, this implies that all degrees in B are at most $|A| = |B'| \leq |B|$. The set of possible

degrees $\{1, 2, \dots, |B|\}$ is covered by the family of $\log_2 |B|$ dyadic intervals $I_k = [2^k, 2^{k+1}]$, so there must be some I_k with the property that at least $|B|/\log_2 |B|$ vertices of B have degrees in I_k . Sample a random subset $A' \subset A$ by taking each vertex independently with probability $p = 2^{-k-1}$, and let B'' be the set of all vertices in B that are adjacent to exactly one vertex in A' . It is clear that $H = A' \cup B''$ is admissible, so it remains to control $|B''|$. Any vertex $v \in B$ has probability exactly $\mathbb{P}[\text{Bin}(d(v), 2^{-k-1}) = 1] = d(v)2^{-k-1}(1 - 2^{-k-1})^{d(v)-1}$ of being chosen for B'' . Since $I_k = [2^k, 2^{k+1}]$, when $d(v) \in I_k$ this probability is bounded from below by an absolute constant (one can take $1/8$). Hence the expected size of B'' is at least $\Omega(|I_k|) \geq \Omega(|B|/\log |B|)$, which implies that there must exist some choice of A' and B'' that satisfy this bound.

The following construction shows that this bound is asymptotically tight. Choose integers $m = 2^k$, let $A = \mathbb{Z}/m\mathbb{Z}$, and let $B = B_0 \cup \dots \cup B_k$, where each $B_i = \{b_{i,1}, \dots, b_{i,m}\}$. Let each $b_{i,j}$ be adjacent to precisely $\{i, i+1, \dots, i+2^j-1\} \in A$, where we reduce everything modulo m . This has $|B| = (k+1)m = \Theta(m \log m)$, but it is not too difficult to verify that any admissible H has $|H \cap B| < 2m$.

3.1. Proof of Lemma 1

Unfortunately, Lemma 2 is insufficient in general for our application, because in our triangle-free graph, the sizes of the connected components V_i of $V \setminus (\{v\} \cup N(v))$ may differ wildly. For this, we need its weighted variant, which we prove in this section. The main trick in the proof is to vary the weights, which leads us to study the following function.

Definition 1. Let G be a bipartite graph with vertex set $A \cup B$. For notational convenience, let the vertices of B be named $\{1, 2, \dots, m\}$. Then, we define

$$F_G(w_1, \dots, w_m) = \max_{\text{admissible } H \subset A \cup B} \sum_{i \in B \cap H} \sqrt{w_i},$$

where we still say that a nonempty subset $H \subset A \cup B$ is *admissible* when every vertex in $B \cap H$ has exactly one neighbor in $A \cap H$.

Lemma 1 is thus equivalent to the statement that $F_G(\mathbf{w}) \geq \sqrt{\sum_{i=1}^m w_i}$ for any collection of $w_i \geq 0$. Since this inequality is homogeneous in the w_i , from now on we will always assume that the weights have been normalized to sum to 1. It then suffices to show that $F_G(\mathbf{w}) \geq 1$ for all \mathbf{w} that satisfy the constraints $w_i \geq 0$, $\sum w_i = 1$. Observe that this domain is now compact, and F_G is a maximum of a finite collection of continuous functions, hence continuous. Therefore, F_G attains its infimum on this domain, which we will denote $\min_{\mathbf{w}} F_G$.

So, suppose for the sake of contradiction that we have some graph G of minimum order for which $\min_{\mathbf{w}} F_G < 1$. Graph G must have an induced matching between A and some subset $B' \subset B$, since otherwise we can use the same reduction argument as in the proof of Lemma 2 to obtain a contradiction to the minimality of G . Let (w_1, \dots, w_m) be a minimizing assignment for F_G , satisfying the constraints $w_i \geq 0$ and $\sum w_i = 1$. Note that actually all w_i must be strictly positive, or else we could delete a vertex i with $w_i = 0$ to obtain a proper induced subgraph G' of G and a weight assignment \mathbf{w}' for which $F_{G'}(\mathbf{w}') < 1$, again contradicting the minimality of G .

We now exploit the fact that we have cast our problem in a continuous setting. Let us study the effect of performing the following perturbation on the weights.

Stage 1. For each $i \in B'$, let $w'_i = w_i - \epsilon \sqrt{w_i}$. For $i \notin B'$, let $w'_i = w_i$.

Stage 2. To compensate for the fact that $\sum w'_i = 1 - \epsilon \sum_{j \in B'} \sqrt{w_j} < 1$, renormalize by scaling up every weight by the same proportion. That is, for all $i \in B$, let

$$w''_i = \frac{w'_i}{1 - \epsilon \sum_{j \in B'} \sqrt{w_j}}.$$

Note that since all $w_i > 0$, for all sufficiently small ϵ , all new w'_i are still positive. This perturbation is chosen in the particular way because for small ϵ and $i \in B'$, $\sqrt{w'_i} = \sqrt{w_i} - \frac{\epsilon}{2} + o(\epsilon)$. This is because it is easy to check that for every $x > 0$,

$$\lim_{\epsilon \rightarrow 0} \frac{\sqrt{x - \epsilon\sqrt{x}} - (\sqrt{x} - \epsilon/2)}{\epsilon} = 0.$$

So, the effect on the square root of each weight w_i with $i \in B'$ is roughly the same, no matter what the weight is.

Now recall that the function $F_G(w_1, \dots, w_m)$ is defined as the maximum over all admissible H of $\sum_{i \in B \cap H} \sqrt{w_i}$. Since all $w_i \geq 0$ by definition, this is equal to the maximum over all maximal admissible H , where this maximality is defined with respect to set inclusion. For brevity, let $M = F_G(w_1, \dots, w_m)$ be that maximum and let H be any such maximal admissible selection.

Note that H must intersect B' (the subset of B that has an induced matching to A). This is because any maximal admissible H contains at least one vertex from A , and that vertex's partner in B' can be added to H while preserving admissibility. In particular, the sum $\sum_{i \in B \cap H} \sqrt{w_i}$ includes at least one downwardly perturbed weight from B' . Therefore,

$$\sum_{i \in B \cap H} \sqrt{w'_i} \leq \left(\sum_{i \in B \cap H} \sqrt{w_i} \right) - \frac{\epsilon}{2} + o(\epsilon) \leq M - \frac{\epsilon}{2} + o(\epsilon).$$

The renormalization that converts w'_i into w''_i is particularly simple to analyze. Using the previous inequality and the observation that $\sum_{j \in B'} \sqrt{w_j} \leq M$ (because $B' \cup A$ is an induced matching, hence admissible):

$$\sum_{i \in B \cap H} \sqrt{w''_i} \leq \frac{M - \frac{\epsilon}{2} + o(\epsilon)}{\sqrt{1 - \epsilon \sum_{j \in B'} \sqrt{w_j}}} \leq \frac{M - \frac{\epsilon}{2} + o(\epsilon)}{\sqrt{1 - \epsilon M}}.$$

The final bound is independent of H , so if it were strictly smaller than M , we would have $F_G(w''_1, \dots, w''_m) < M = F_G(w_1, \dots, w_m)$, contradicting the minimality of (w_1, \dots, w_m) . Therefore, we must have:

$$\begin{aligned} \frac{M - \frac{\epsilon}{2} + o(\epsilon)}{\sqrt{1 - \epsilon M}} &\geq M, \\ M - \frac{\epsilon}{2} + o(\epsilon) &\geq M\sqrt{1 - \epsilon M}, \\ M - \frac{\epsilon}{2} + o(\epsilon) &\geq M\left(1 - \frac{\epsilon M}{2} + o(\epsilon M)\right), \\ -\frac{\epsilon}{2} + o(\epsilon) &\geq -\frac{\epsilon M^2}{2} + o(\epsilon M^2), \\ 1 - o(1) &\leq M^2. \end{aligned}$$

In the final inequality, we used the fact that M is fixed, and therefore $o(M^2) = o(1)$. Sending ϵ to zero, we conclude that $F_G(\mathbf{w}) = M \geq 1$. This contradicts our assumption that $F_G(\mathbf{w}) = \min_{\mathbf{w}} F_G < 1$, so our proof is complete.

Remark. The following example shows that the assertion of Lemma 1 no longer holds for any exponent $\alpha > 1/2$. Indeed, consider the following bipartite graph. For some very large t , let $A = \{a_1, \dots, a_t\}$, let $B = \{b_0, \dots, b_t\}$, and connect each a_i to b_0 and b_i . Let the weight of b_0 be $1 - t^{-1}$, and the weights of all other vertices in B be t^{-2} , so the total weight is 1. It is easy to see that the only maximal admissible sets in this graph are either a star containing b_0 and some other b_i , or the induced matching between A and $B \setminus \{b_0\}$. Since $\alpha > 1/2$ and t is sufficiently large, we have in the first case that $(1 - t^{-1})^\alpha + (t^{-2})^\alpha = 1 - \alpha t^{-1} + o(t^{-1}) < 1$. On the other hand, for the second admissible set, we only have $t \cdot (t^{-2})^\alpha = t^{1-2\alpha} < 1$.

4. K_r -free graphs

This section is devoted to the proof of Theorem 2. The induction approach we used in Section 2 easily extends to the case of K_r -free graphs when $r \geq 4$, and in fact the argument becomes even simpler. We prove that for any $r \geq 4$, every connected K_r -free graph $G = (V, E)$ with $n + 1$ vertices contains an induced tree of size $\frac{\log n}{4 \log r} + 1$ through any particular vertex. Note that since the logarithm appears in both the numerator and denominator, its base is irrelevant. The statement is clearly true for $n = 1$, which starts our induction.

Now, consider any $n \geq 2$, and suppose that the statement holds for all smaller values of n . Let $v \in V$ be an arbitrary vertex. We will find an induced tree of size $\frac{\log n}{4 \log r} + 1$ containing v . Recall the well-known fact from Ramsey Theory (see, e.g., Chapter 6.1 of [3]) that any graph with $a^b \geq \binom{a+b-2}{a-1}$ vertices contains either a clique of size a or an independent set of size b . This implies that the degree of v must be less than $r^{\log n / 4 \log r} = n^{1/4}$, or else we would already be done. Indeed, since G is K_r -free, the neighborhood of v would then contain an independent set of size $\frac{\log n}{4 \log r}$. The vertices of this set together with v form an induced star of the desired size. The same argument also shows that every $w \in N(v)$ has less than $n^{1/4}$ neighbors in $V \setminus (\{v\} \cup N(v))$. Otherwise, by the above discussion, we could find an independent set $I \subset V \setminus (\{v\} \cup N(v))$ of size $\frac{\log n}{4 \log r}$, all of whose vertices are adjacent to w . Then, v, w , and I will form a large induced tree containing v .

Let V_1, \dots, V_m be the vertex sets of the connected components of the subgraph of G induced by $V \setminus (\{v\} \cup N(v))$. Since G is connected, each V_i is adjacent to some vertex in $N(v)$. As we explained above, each vertex in $N(v)$ is adjacent to fewer than $n^{1/4}$ sets V_i , so in particular $m < |N(v)|n^{1/4} < n^{1/2}$. We claim that all components V_i have size at most $\frac{n}{r^4}$. Indeed, suppose that some $|V_i|$ exceeds $\frac{n}{r^4}$. Let u be a vertex in $N(v)$ which is adjacent to at least one vertex in V_i . Applying the induction hypothesis to $\{u\} \cup V_i$, we find an induced tree T through u of size $\frac{\log(n/r^4)}{4 \log r} + 1 = \frac{\log n}{4 \log r}$. Then $\{v\} \cup T$ gives an induced tree of the desired size.

Next, we show that there are more than r^2 indices i for which $|V_i| \geq \frac{\sqrt{n}}{r^2}$. Indeed, if this were not the case, then the total number of vertices in V would be less than:

$$|\{v\} \cup N(v)| + \sum_{i=1}^m |V_i| < 1 + n^{1/4} + m \cdot \frac{\sqrt{n}}{r^2} + r^2 \cdot \frac{n}{r^4} < 1 + n^{1/4} + 2 \cdot \frac{n}{r^2} \leq 1 + n^{1/4} + \frac{n}{8}.$$

This is less than $n + 1 = |V|$ for all $n \geq 2$, which is a contradiction.

Let B be the above set of indices for which $|V_i| \geq \frac{\sqrt{n}}{r^2}$, and let $A = N(v)$. Consider the auxiliary bipartite graph with sides A and B obtained by connecting $u \in A$ with $i \in B$ if u is adjacent to at least one vertex in V_i . Applying Lemma 2, we find subsets $A' \subset A$ and $B' \subset B$ with $|B'| \geq \sqrt{|B|} > r$ such that for each $i \in B'$ the component V_i is adjacent to exactly one vertex in $A' \subset N(v)$, which we denote $u(i)$. In fact, $|B'| \geq r + 1$ since both $|B'|$ and r are integers. Apply the induction hypothesis to each $\{u(i)\} \cup V_i$ to find an induced tree T_i containing $u(i)$ of size at least $\frac{\log |V_i|}{4 \log r} + 1$.

If all $u(i)$ are distinct, then we can find $u(i) \neq u(j)$ which are not adjacent in G , because this is a set of at least $r + 1$ vertices in a K_r -free graph. Then, $\{v, u(i), u(j)\} \cup T_i \cup T_j$ is an induced tree. On the other hand, if there is some $u(i) = u(j)$, then $\{v, u(i)\} \cup T_i \cup T_j$ is an induced tree. In either case, we find an induced tree containing v of size at least

$$|\{v, u(i), u(j)\}| + \frac{\log |V_i|}{4 \log r} + \frac{\log |V_j|}{4 \log r} \geq 2 + 2 \cdot \frac{\log(\sqrt{n}/r^2)}{4 \log r} = 1 + \frac{\log n}{4 \log r},$$

as desired. This completes the proof.

5. Concluding remarks

In this paper, we obtain a lower bound on the size of the largest induced-tree in a K_r -free graph, which is tight up to a small multiplicative constant. Moreover, our proof shows that we can find a large tree through any particular vertex in the graph. It turns out that this seemingly stronger result is equivalent up to a constant factor to the original problem of finding one large tree.

Claim. Let T be an induced tree in a connected graph G , and let v be an arbitrary vertex. Then G has an induced tree of size $1 + \frac{|T|}{2}$ which contains v .

Proof. If v is already in T , then there is nothing to prove. Otherwise, let $P = (v_1, v_2, \dots, v_m)$ be a shortest path between v and T , with $v_1 = v$ and $v_m \in T$. By minimality of P , there are no edges between $\{v_1, \dots, v_{m-2}\}$ and T . Let e_1, \dots, e_k be the edges connecting v_{m-1} and T , and let t_1, \dots, t_k be their endpoints in T . Since T is a tree, by deleting some edges, we can partition it into subtrees T_1, \dots, T_k , such that each T_i contains t_i . Consider the auxiliary graph on k vertices, in which vertices i and j are adjacent if there is an edge of G between T_i and T_j . Note that this graph also forms a tree, and therefore is bipartite. Hence, we can find two disjoint subsets $I \cup J = [k]$ such that the collection of T_i with $i \in I$ has no edges crossing between them, and similarly, the collection of T_j with $j \in J$ also has no crossing edges. Therefore, the union of $\{v_1, \dots, v_{m-1}\}$ with either one of these two collections will form an induced tree. Clearly, both of these trees contain v , and their union covers T . Thus, one of them has size at least $1 + \frac{|T|}{2}$. \square

We also wish to remark that for the problem of finding a large induced tree through every vertex of a triangle-free graph, one can improve the $2\sqrt{n}$ upper bound of Matoušek and Šámal. Indeed, consider the following triangle-free graph on n vertices. Let $m = \sqrt{2n}$, and take the graph with parts V_0, \dots, V_{m-1} , where $V_0 = \{v\}$, every other $|V_i| = m - i$, and each consecutive pair of parts (V_i, V_{i+1}) induces a complete bipartite graph. This is a bipartite (hence also triangle-free) graph with $1 + m(m-1)/2 = (1 + o(1))n$ vertices, but one can easily check that any induced tree containing v has at most $m = \sqrt{2n}$ vertices. In particular, this shows that any approach which guarantees a large tree through every vertex of the graph cannot match Matoušek and Šámal's upper bound.

In light of this discussion, we do not have a clear conjecture as to what is the right constant in front of \sqrt{n} in the problem of finding a maximum induced tree in a triangle-free graph on n vertices. Nevertheless, as the upper and lower bounds are now so close, perhaps there is a hope to bridge this gap with other methods.

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