Graph theory

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1 Problems and famous results

1. (Putnam 1957/A5.) Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.

   Solution: Show that if there is any vertex of degree ≥ 3 in the unit distance graph, then it has a neighbor of degree 1 in the unit distance graph. Pulling off that neighbor by induction solves the problem, or else all degrees are ≤ 2, at which point the edge bound follows.

2. Every tournament (complete graph with each edge oriented in some direction) contains a Hamiltonian directed path (hitting every vertex exactly once).

3. (Romania 2006.) Each edge of a polyhedron is oriented with an arrow such that every vertex has at least one edge directed toward it, and at least one edge directed away from it. Show that some face of the polyhedron has its boundary edges coherently oriented in a circular direction.

   Solution: A directed cycle in the graph exists by simply following out-edges until we repeat vertices. If it has stuff inside it, then one can cut the cycle with a directed path, and then there is a shorter directed cycle. Compare the sizes of cycles by the number of faces they contain.

4. (Monotone paths.) Show that for every even n, it is possible to label the edges of K_n with the distinct integers $1, 2, \ldots, \binom{n}{2}$, in such a way that no increasing walk contains more than $n - 1$ edges. An increasing walk is a sequence of vertices $v_0, v_1, \ldots, v_t$ such that the labels of the edges $v_i v_{i+1}$ increase with i. The vertices $v_0, \ldots, v_t$ are not required to be distinct—that is the difference between the definitions of walks and paths.

5. (Sweden 2010.) A town has $3n$ citizens. Any two persons in the town have at least one common friend in this same town. Show that one can choose a group consisting of n citizens such that every person of the remaining $2n$ citizens has at least one friend in this group of n.

   Solution: The codegree condition implies that the diameter of the graph is at most 2. We prove that every n-vertex graph with diameter ≤ 2 has a dominating set (a subset S of vertices such that every other vertex is either in, or has a neighbor in S) of size only $\leq \sqrt{n \log n} + 1$. To see this, let $p = \sqrt{\frac{\log n}{n}}$.

   Observe that since the diameter is at most 2, if any vertex has degree $\leq np$, then its neighborhood already is a dominating set of suitable size. Therefore, we may assume that all vertices have degree strictly greater than np. It feels "easy" to find a small dominating set in this graph because all degrees are high. Consider a random sample of np vertices (selected uniformly at random, with replacement), and let S be their union. Note that $|S| \leq np$. Now the probability that a particular fixed vertex v fails to have a neighbor in S is strictly less than $(1-p)^{np}$, because we need each of np independent samples to miss the neighborhood of v. This is at most $e^{-np^2} \leq e^{-\log n} = n^{-1}$. Therefore, a union bound over the n choices of v produces the result.

6. (Bondy 1.5.9.) There are n points in the plane such that every pair of points has distance ≥ 1. Show that there are at most $3n$ (unordered) pairs of points that span distance exactly 1 each.

   Solution: The unit distance graph is planar.

7. (Prüfer.) A graph with vertex set $\{1, \ldots, n\}$ is a *spanning tree* if it is a tree which includes all of those n vertices. It turns out that there is a surprisingly beautiful formula for the number of spanning trees on $\{1, \ldots, n\}$: it is just n^{n-2}.

8. Let n be an even integer. It is possible to partition the edges of K_n into exactly $n-1$ perfect matchings. (In this context of non-bipartite graphs, a perfect matching is a collection of $n/2$ edges that touch every vertex exactly once.) We can interpret that as a way to run a round-robin sports tournament among n teams: on each of $n-1$ days, the n teams pair up according to the day's perfect matching, and each of the $n/2$ edges tells who plays who that day. There are $n/2$ simultaneous games on each of the $n-1$ days.

   On each of the $n-1$ days, there are $n/2$ winning teams from the $n/2$ games. So, there are $n/2$ winners of Day 1, $n/2$ winners of Day 2, $\ldots$, and $n/2$ winners of Day $(n-1)$. Prove that no matter how the $\binom{n}{2}$ individual games turned out, it is always possible (after all of the games) to select one team who was a winner of Day 1, one team who was a winner of Day 2, $\ldots$, and one team who was a winner of Day $(n-1)$, such that we don't pick the same team twice. Note that since there are n teams in total, this selection will always leave exactly one team out.