Graph theory

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1 Problems and famous results

1. (Sweden 2010.) A town has $3n$ citizens. Any two persons in the town have at least one common friend in this same town. Show that one can choose a group consisting of n citizens such that every person of the remaining $2n$ citizens has at least one friend in this group of n.

   Solution: The codegree condition implies that the diameter of the graph is at most 2. We prove that every n-vertex graph with diameter ≤ 2 has a dominating set (a subset S of vertices such that every other vertex is either in, or has a neighbor in S) of size only $\leq \sqrt{n \log n} + 1$. To see this, let $p = \sqrt{\frac{\log n}{n}}$.

   Observe that since the diameter is at most 2, if any vertex has degree $\leq np$, then its neighborhood already is a dominating set of suitable size. Therefore, we may assume that all vertices have degree strictly greater than np. It feels "easy" to find a small dominating set in this graph because all degrees are high. Consider a random sample of np vertices (selected uniformly at random, with replacement), and let S be their union. Note that $|S| \leq np$. Now the probability that a particular fixed vertex v fails to have a neighbor in S is strictly less than $(1-p)^{np}$, because we need each of np independent samples to miss the neighborhood of v. This is at most $e^{-np^2} \leq e^{-\log n} = n^{-1}$. Therefore, a union bound over the n choices of v produces the result.

2. (Erdős-Rényi graph.) There is an n-vertex graph with all degrees of order $\Theta(\sqrt{n})$, which has diameter 2.

   Solution: Take the polarity graph of the projective plane. Suppose that p is a prime, and identify the vertices with the 1-dimensional subspaces of $\mathbb{F}_p^3$. Put an edge if two 1-dimensional subspaces are orthogonal. The diameter is 2 because given any two 1-dimensional subspaces, their orthogonal complement is 1-dimensional, and orthogonal to both. To bound the degrees, note that the orthogonal complement of a fixed 1-dimensional subspace is a 2-dimensional subspace, and the number of distinct 1-dimensional subspaces in it is exactly $\frac{p^2-1}{p-1} = p+1$. (If this orthogonal complement includes the original vector, i.e., when it is self-orthogonal, then the degree will actually only be p.) Yet the total number of vertices is the number of 1-dimensional subspaces of $\mathbb{F}_p^3$, i.e., $\frac{p^3-1}{p-1} = p^2+p+1$, which is about the square of the degree.

3. (Belarus 2010/C7.) Let $n \geq 3$ distinct points be marked on a plane so that no three of them lie on the same line. All points are connected with the segments. All segments are painted one of the four colors so that if in some triangle (with the vertices at the marked points) two sides have the same color, then all its sides have the same color (each of the four colors is used). What is the largest possible value of n?

   Solution: Answer: 9. This makes each color class an equivalence relation, so that each color class is the vertex-disjoint union of cliques. Consider a maximal clique in color 1. It can't be everybody, so there's someone else, v. Since it's maximal, every edge from v to it has color not 1. But equivalence classes, so the edges from v to it are all different colors. There are only 3 colors left. Thus the maximal clique has order at most 3. In particular, every color class's cliques are at most triangles, and hence the maximum degree in each color class is at most 2.

There are only 4 colors, so the maximum degree is at most 8, and thus $n \leq 9$. Construction for $n = 9$: split the vertices into three triples for the first color class: (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3). Next color class makes 3 cliques (a_i, b_i, c_i) for constant i. Next color class makes 3 cliques (a_i, b_{i+1}, c_{i+2}), wrapping mod 3. Fourth color class makes 3 cliques (a_i, b_{i+2}, c_{i+4}), wrapping mod 3.

4. (Turán.) For every n and r, the n-vertex graph with the maximum number of edges, but no K_r-subgraph, is the complete $(r-1)$-partite graph in which all part sizes are as equal as possible.

5. (Kirkman, Query VI in The Lady's and Gentleman's Diary.) Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

6. (Ray-Chaudhuri, Wilson.) This is always possible for n girls, where n is an odd multiple of 3, and they walk in threes for $\frac{n-1}{2}$ days.

7. (Affine plane.) For every prime p, it is possible to partition the edges of K_{p^2} into disjoint copies of K_p.

   Solution: Identify p^2 with the two-dimensional $\mathbb{F}_p^2$. For There are many lines of the form $y = ax + b$, for $a, b \in \mathbb{F}_p$, and also add the vertical lines $x = c$. Every pair of distinct lines intersects at either 0 or 1 points, so if we make each line a clique, then we have the desired partition.

8. (Wilson.) For every graph H, there is an integer N such that whenever $n > N$, $\binom{n}{2}$ is divisible by $e(H)$, and $(n-1)$ is divisible by the GCD of the degrees of H, then the edges of K_n can be partitioned into disjoint copies of H.

9. (Dirac.) If an n-vertex graph has all degrees at least $n/2$, then it has a Hamilton cycle, i.e., a cycle which uses every vertex exactly once.

10. (Sweden 2010.) Some of n students in a class ($n \geq 4$) are friends. Any $n-1$ students in the class can form a circle so that any two students next to each other on the circle are friends, but all n students cannot form a similar circle. Find the smallest possible value of n.

    Solution: Answer: $n = 10$, the Petersen graph. It is well-known that the Petersen graph is not Hamiltonian, but it is easy to see that if one deletes any vertex, one can find a Hamilton cycle. This is easy to check by symmetry, because all outer vertices are the same, and all inner vertices are the same.

    To see why Petersen is not Hamiltonian, observe that it is two disjoint 5-vertex graphs linked by a single perfect matching. Any Hamilton cycle must cross back and forth between the parts. If it just goes across once, and then back, then on each side it must visit all 5 vertices in one go. Those are paths of length 4, and it's easy to see that if one takes 4 consecutive edges along the outer cycle, then it doesn't complete to an H-cycle. Otherwise, the H-path must go across, back, across, and back. It can't do more times because there are only 5 matching edges. In this case, WLOG start with two consecutive outer edges, and take forced moves until we are stuck without an H-cycle.

    Now we must show that no $n \leq 9$ will work. First key observation: if there is a vertex of degree 2 or less, then it's impossible. Indeed, if so, then delete a neighbor of it; the remainder must be Hamiltonian, but now this vertex has degree ≤ 1, contradiction. Thus the minimum degree is at least 3, which already disposes of all cases $n \leq 6$.

    Next observation: let v be the max-degree vertex. Since $G - v$ is still Hamiltonian, take an H-cycle of the remainder. If two adjacent vertices of the cycle are neighbors of v, then we can extend, contradiction. So v's neighbors on the cycle are separated by at least one vertex each.

    For $n = 7$, we can't have all degrees equal to 3, because the sum of degrees must be even. Thus there is a degree-4 vertex. But the remainder cycle has 6 vertices, so it's impossible to alternate. (Another way to see that $n = 7$ fails: using an H-cycle from $G - v$, and then adding v, we get an H-path with v as an endpoint, but the other endpoint has degree ≥ 3, and $4+3 \geq n$, so the Ore-type condition wins.)

For $n = 8$, the above alternating condition shows that it is over if there is a vertex of degree 4. Hence the graph is 3-regular. Pull out a vertex, and look on the remaining 7-cycle. Add back its neighbors, and there is actually only one way to do so with proper spacing. Then there are only 2 ways to complete to a 3-regular graph, and in both cases the entire thing is Hamiltonian.

For $n = 9$, we can't have all odd degrees, so there's a vertex of degree 4. It must interact with the remaining 8-cycle in exactly one way: alternating neighbors. Now the rest of the vertices on the 8-cycle need degrees ≥ 3. Let A be the set of neighbors of v, and let B be the others. If two consecutive (separated only by a single vertex of A) vertices of B are adjacent, then one can see that the whole thing is Hamiltonian. If two opposite vertices of B are adjacent, then also the whole thing is Hamiltonian. Thus the only way to relieve the degrees is to have the vertices of B adjacent to vertices of A that they are not already adjacent to. And we can't create any vertices of degree 5, or else done by failing to alternate as above. Then there is exactly one way to make this graph, and already, all vertices of A have degree 4 and all vertices of B have degree 3, and v has degree 4. No more edges can be added because connecting two vertices of B wins already as above. Thus this is the whole graph. But deleting any vertex of A we can see that the remainder is not Hamiltonian.

11. (Hungary 2010.) Prove that the edges of the complete graph with 2009 vertices can be labeled with $1, 2, \ldots, \binom{2009}{2}$ such that the sum of the labels corresponding to all edges having a given vertex is different for any two vertices.

Solution: Let $n = 2009$. Consider the random labeling. Let u and v be two fixed vertices. The label of the edge between u and v is irrelevant for the bad event that the label sums are equal at u and v. Expose the $n-2$ labels of edges from u to $[n] \setminus \{u, v\}$. Let their sum be S. Next we want to expose the $n-2$ labels of the edges from v to $[n] \setminus \{u, v\}$. It suffices to show that conditioned on the $n-2$ labels we already saw from u, their sum equals S with probability less than $1/\binom{n}{2}$, because then a union bound implies that there is a labeling that avoids all bad events.

Intuitively, the probability is actually of order $n^{-5/2}$. To see this, suppose that the $n-2$ new random labels are sampled independently with replacement from the full set $I = \{1, \ldots, \binom{n}{2}\}$. If we sample one integer from I, the variance is of order n^4. Therefore, the variance of this slightly different sum is of order n^5, and since we are adding i.i.d. random variables, the distribution is "nice", and the probability that the sum is any particular number is of order at most $n^{-5/2}$.

Now we formalize this. We make exactly $n-3$ i.i.d. samples from I. After this, we look to see whether we ever got the same label multiple times, or if we repeated a label we saw from u. For each of these occurrences, we re-sample uniformly from I until we find new labels, and ultimately build a set of $n-3$ distinct new labels. Finally, we repeat this procedure until we get a final new label, and that produces a set of $n-2$ labels which are distinct from those from u, while also uniformly distributed over all possibilities.

The first observation is that during the first round, the probability that we hit a repeat label is less than $(2n-2)/\binom{n}{2} = \frac{4}{n}$. Therefore, the number of times we will have to resample in the second round is stochastically dominated by Bin $\left[n, \frac{4}{n}\right]$, and the probability that such a Binomial exceeds $\log n$ is at most

$$\binom{n}{\log n} \left(\frac{4}{n}\right)^{\log n} < \left(\frac{4e}{\log n}\right)^{\log n} \ll n^{-3}.$$

Now note that success (getting a sum of exactly S) comes in one of two ways: (1) if the Binomial exceeds $\log n$, and then we get lucky, or (2) the Binomial stays below $\log n$, the sum of the $n-3$ labels after the first round is within $n^2 \log n$ of S, and then after the second round, the final label in the third round makes the sum exactly S. The chance of winning from (1) is at most n^{-3} from above. The chance of winning from (2) is at most the probability that the sum of the $n-3$ labels after the first round is within $n^2 \log n$ of S, and the final label makes the sum exactly S.

We calculate this by multiplying upper bounds of the probabilities that (a) the first round sum is within $n^2 \log n$ of S and (b) the conditional probability that the third round label makes the sum exactly equal to S. The latter probability is obviously at most $\frac{3}{n^2}$ because there is only one choice for it which would make the sum S. The probability of (a) can be bounded by the Central Limit Theorem, because the first round sum is precisely the sum of i.i.d. random variables with bounded second moment. So, by the CLT, the probability that this sum lies within any given window of at most $n^2 \log n$, given that the variance of the sum should be of order n^5, is $o(1)$.

12. Find a 3-edge coloring of the dodecahedron.

    Solution: Find a Hamilton cycle. The remainder is 1-regular, hence a pefect matching. Alternately color the edges of the H-cycle.

13. (Vizing.) If the maximum degree of a graph is Δ, then at least Δ colors are required to properly color its edges, but $\Delta + 1$ colors are sufficient.