Graph Theory II

Po-Shen Loh

June 2009

1 Warm-up

1. Let n be odd. Partition the edge set of K_n into n matchings with $\frac{n-1}{2}$ edges each.

   Solution: Spread the n vertices around a circle. Take parallel classes.

2. Let n be even. Partition the edge set of K_n into $n-1$ matchings with $\frac{n}{2}$ edges each.

   Solution: Spread $n-1$ vertices around a circle, and let the final vertex be the origin. Take parallel classes, along with the orthogonal radius from the origin.

3. (Bondy 1.5.9.) There are n points in the plane such that every pair of points has distance ≥ 1. Show that there are at most $3n$ (unordered) pairs of points that span distance exactly 1 each.

   Solution: The unit-distance graph is planar. If there are 2 crossing unit-distance edges, then the triangle inequality implies that some pair of points is at distance strictly less than 1.

2 Matching

Consider a bipartite graph $G = (V, E)$ with partition $V = A \cup B$. A *matching* is a collection of edges which have no endpoints in common. We say that *A has a perfect matching to B* if there is a matching which hits every vertex in A.

Theorem. *(Hall's Marriage Theorem) For any set $S \subset A$, let $N(S)$ denote the set of vertices (necessarily in B) which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for **every** $S \subset A$.*

This has traditionally been called the "marriage" theorem because of the possible interpretation of edges as "acceptable" pairings, with the objective of maximizing the number of pairings. In real life, however, perhaps there may be varying degrees of "acceptability." This may be formalized by giving each vertex (in both parts) an ordering of its incident edges. Then, a matching M is called *unstable* if there is an edge $e = ab \notin M$ for which both a and b both prefer the edge e to their current partner (according to M).

Theorem. *(Stable Marriage Theorem) A stable matching always exists, for every bipartite graph and every collection of preference orderings.*

Now try these problems.

1. Let G be a bipartite graph with all degrees equal to k. Show that G has a perfect matching.

   Solution: Fix any set X, and consider $N(X)$. The number of edges coming out of X is exactly $k|X|$, but each vertex in $N(X)$ can only absorb $\leq k$ many of them. So $|N(X)| \geq |X|$, and we have Hall's condition.

2. (J. Hirata.) Suppose that a regular deck of 52 playing cards has been dealt into 13 piles of 4 cards each. Show that there is a way to select one card from each pile, such that you have one card from every rank (ace, 2, ..., king).

   Solution: Consider the bipartite graph where LHS corresponds to the 13 piles, and RHS corresponds to the 13 ranks. Put an edge from a pile to a rank if the pile contains at least one card of that rank.

   At first this appears to be a special case of the previous problem with $k = 4$, but now we have a problem of multiple edges: a pile might be all kings, for example.

   However, the same argument works. For the purpose of bookkeeping only, write weights on the edges. If there are two aces in a pile, for example, let the corresponding edge have weight 2. We will still search for a perfect matching in the usual sense (disregarding weights).

   Fix some set X from LHS, and count the total weight emanating from it to $N(X)$. This is exactly $4|X|$. Again, each guy in $N(X)$ can absorb only ≤ 4 of this weight, so we again have $|N(X)| \geq |X|$.

3. Let A be a square $n \times n$ matrix of nonnegative integers, in which each row and column sum up to the positive integer m. Prove that A can be expressed as a sum of m *permutation matrices* $A = P_1 + \cdots P_m$. Here, a permutation matrix is an $n \times n$ matrix of zeros and ones, such that each row contains a single "1", and each column contains a single "1."

   Solution: Pull out permutation matrices inductively via Hall's theorem. LHS is the rows, RHS is the columns, and there is an edge between i on the LHS and j on the RHS if a_{ij}, the (i, j) entry of A, is > 0. Actually, write the number a_{ij} next to that edge.

   For any subset S of the LHS, note that the sum of all numbers on edges coming out of S is precisely $m|S|$. But those edges are a subset of the edges coming out of $N(S)$, so their sum is $\leq m|N(S)|$. Thus $|N(S)| \geq |S|$ for any S.

4. (Gale-Shapley stable marriage protocol.) Suppose there are equal numbers of men and women, and each person ranks the opposite group according to preference. A series of rounds ensue. In each round:

   - Every currently-not-engaged man simultaneously proposes to the top choice among those remaining on his list.
   - Then each woman compares her incoming proposals to the person she is currently engaged to. She selects the best of these (possibly dumping her current engagement), and is now engaged to the new person.
   - All rejected men permanently remove the corresponding women from their list.

   Show that when this process terminates, everybody has gotten married. Next, show that the resulting marriage is stable.

   Solution: First observe that if a woman is *ever* proposed to, then she will be married at the end of the procedure, because men can never break engagements.

   Now suppose for contradiction that after the entire protocol, some man M is still unmarried. Since there were equal numbers of men and women, there is also an unmarried woman W. At some point, M should have proposed to W. But we started by showing that if W was ever proposed to by anybody, she should end up married, contradicting the fact that W is still not married.

   Next, we show that this marriage is stable. For contradiction, suppose there are some M and W who are not married to each other, but M actually prefers W over his current wife W', and W actually prefers M over her current husband M'. In that case, M should have proposed to W before proposing to W', but must have been rejected by W. But women only improve their matches throughout this procedure, so this implies that W actually prefers her current M' over M, contradiction.

5. (Turkey 1998/4.) To n people are to be assigned n different houses. Each person ranks the houses in some order (with no ties). After the assignment is made, it is observed that every other assignment

assigns at least one person to a house that person ranked lower than in the given assignment. Prove that at least one person received his/her top choice in the given assignment.

Solution: This proof is similar to the existence of a stable marriage. Pick a guy, call it x_1, free up his assignment (call it y_0), and steal his top pick, call that y_1. But now somebody, say x_2, lost their assignment. If x_2's top assignment was y_0, then everyone would be happier if x_2 just took y_0 now. That would be a contradiction.

So, x_2 has a top pick somewhere, say y_2. Note that $y_2 \neq y_1$ or else x_2 was already assigned to his favorite. Steal that from, say, x_3, and repeat. We never will end up with any x_i having y_0 as his top pick, or else we immediately have improved everybody.

But then at some point we will loop since everything is finite. This means that some x_t will have stolen some y_k which we had previously reassigned to some x_k. That's fine—undo all reassignments of x_1 through x_k, so that they will match back to their original y_0 through y_{k-1}. We now have a perfect matching again, but everybody from x_{k+1} through x_t in fact improved all the way up to their top choice, contradiction.

6. (Diestel 2.7.) Let $S = \{1, 2, \ldots, kn\}$, and suppose $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are both partitions of S into n sets of size k. Then there exists a set T of size n such that every intersection $T \cap A_i$ and $T \cap B_i$ has cardinality exactly 1.

   Solution: Let the LHS have n vertices corresponding to the A_j, and the RHS have n vertices corresponding to the B_j. Consider any $i \in S$. Since we have two partitions, i is in exactly one A-set and exactly one B-set. Say they were A_a and B_b. Then, put an edge between the corresponding vertices in the graph, and label it with i.

   Note that any perfect matching will give a set T: just let T be the collection of the labels of the edges in the matching. To see that it exists, consider any subset S of LHS, and note that since all sets have size exactly k, the number of edges coming out of S is precisely $k|S|$. But this is a subset of the edges coming out of $N(S)$, so $k|S| \le k|N(S)|$, done.

3 Planarity

When we represent graphs by drawing them in the plane, we draw edges as curves, permitting intersections. If a graph has the property that it can be drawn in the plane without any intersecting edges, then it is called *planar*. Here is the tip of the iceberg. One of the most famous results on planar graphs is the Four-Color Theorem, which says that every planar graph can be properly colored using only four colors. But perhaps the most useful planarity theorem in Olympiad problems is the Euler Formula, which is the first item below.

Try these problems.

1. (Euler Formula.) Every connected planar graph satisfies $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces. You may assume that the curves which connect pairs of vertices are piecewise-linear.

   Solution: Actually prove that $V - E + F = 1 + C$, where C is the number of connected components. Each connecting curve is piecewise-linear, and if we add vertices at the corners, this will keep $V - E$ invariant. Now we have a planar graph where all connecting curves are straight line segments.

   Then induction on $E + V$. True when $E = 0$, because $F = 1$ and $V = C$. If there is a leaf (vertex of degree 1), delete both the vertex and its single incident edge, and $V - E$ remains invariant. If there are no leaves, then every edge is part of a cycle. Delete an arbitrary edge, and that will drop E by 1, but also drop F by 1 because the edge was part of a cycle boundary, and now that has merged two previously distinct faces.

2. If G is a planar graph, then $E \leq 3V$.

   Solution: Break into connected components. Special case is the components with only 1 or 2 vertices, but still true in those cases. Otherwise, for each face, calculate its perimeter, and add all of these up. This double-counts each edge. Each face has perimeter ≥ 3, so we get $2E \geq 3F$. Plugging in, we have

   $$2 = V - E + F \leq V - E + \frac{2}{3}E = V - \frac{1}{3}E.$$

3. Every planar graph contains a vertex of degree at most 5.

   Solution: Actually use the bound $E \leq 3V - 6$ for $V \geq 3$, and if $V \leq 2$ it is trivial.

4. Show that K_5 is not planar.

   Solution: $V = 5$, $E = 10$, so we must have $F = 2 - V + E = 7$. But as in the previous solution, we need to have $2E \geq 3F$, which is not the case.

5. Show that $K_{3,3}$ is not planar.

   Solution: $V = 6$, $E = 9$, so we must have $F = 2 - V + E = 5$. But as in the previous solution, we need to have $2E \geq 3F$. Actually, we need $2E \geq 4F$, because $K_{3,3}$ has no triangles. But this stronger inequality is false.

6. Show that $K_{4,4}$ is not planar.

   Solution: $K_{4,4}$ contains $K_{3,3}$.

4 Harder problems

Since the level appears to vary widely, especially in the Blue group, this section collects more challenging questions to work on if you are ahead during lecture.

1. (Petersen's Theorem.) Let G be a graph with all degrees equal to $2k$. Show that one can select a subset F of edges such that every vertex of G is incident to exactly 2 edges in F.

   Solution: Take an Eulerian tour of G. Now duplicate each vertex v into v^- and v^+. If the Eulerian tour goes $\overrightarrow{uw}$, then put an edge from u^- to w^+. This makes a bipartite graph $G^{(2)}$ with all degrees exactly k, so there is a perfect matching. Collapsing each $\{v^+, v^-\}$, we get a 2-factor of the original G.

2. (Optimality in Stable marriage protocol.) Let there be equal numbers of men and women. We know that a stable marriage exists. For each man, let his *realm of possiblity* be the set of all women that he could be married to, going over all stable marriages. Say that his *optimal mate* is his favorite choice from his realm of possibility, and his *pessimal mate* is his least-favorite choice from his realm of possibility. Define these things similarly for each woman.

   According to the Gale-Shapley protocol, men have the first choice in proposing, but can often be rejected because women have the final say in everything. So does anybody have an advantage? Decide whether men or women always get their optimal or pessimal mates according to this protocol.

   Solution: We first show that men always get their optimal mate. Proceed by contradiction, and suppose that M is the *first* man to be rejected by his optimal mate W. Rejection means that W selected some other M' who had proposed to her at the same time or earlier. *First* also means that M' was not yet rejected by his optimal mate, which means that he prefers W at least as much as his optimal mate.

   Since W is M's optimal mate, there is some stable marriage that has them married, and this must have M' married to some W'. But now $M'W$ would rather get married. Indeed, we said above that

W prefers M' over M, and also that W was at least as good as M''s optimal mate. But optimal mate is the best over all stable marriages, so W is also at least as good as W'. Since $W \neq W'$ and there are no ties in preference lists, M' actually prefers W over W', and this contradicts stability.

Next we show that male-optimality implies that every woman gets her pessimal mate. Consider some woman W, and suppose that the male-optimal protocol pairs her with M. Now consider any other stable marriage in which W is paired with someone else, say M', and M must also be paired with someone else, say W'. It suffices to show that W prefers M' over M. By male-optimality of the first protocol, M must prefer W over W'. But we supposed this was a stable marriage, so W must prefer her current mate (M') over M. This completes the proof.

3. (Dirac's Theorem.) Let G be a graph on n vertices with all degrees at least $n/2$. Show that G has a Hamiltonian cycle.

   Solution: Suppose the longest path has t vertices $x_1, \ldots, x_t$. We will show there is a cycle of t vertices as well. Suppose not. All neighbors of x_1 and x_t must lie on the path or else it is not longest. Minimum degree condition implies that both have degree $\geq t/2$. But if $x_1 \sim x_k$, then $x_t \not\sim x_{k-1}$ or else we can re-route to get a cycle. So, each of x_1's $t/2$ neighbors on the path prohibit a potential neighbor of x_t. Yet x_t's neighbors come from indices $1 \ldots t-1$, so there is not enough space for x_t to have $t/2$ neighbors there, avoiding the prohibited ones.

   Now if this longest path is not the full n vertices, then we get a cycle C missing some vertex x. But min-degree $n/2$ implies that the graph is connected (smallest connected component is $n/2+1$), so there is a shortest path from x to C, and adding this to the cycle gives a longer path than t, contradiction.

4. (St. Petersburg 1997/13.) The sides of a convex polyhedron are all triangles. At least 5 edges meet at each vertex, and no two vertices of degree 5 are connected by an edge. Prove that this polyhedron has a face whose vertices have degree 5, 6, 6, respectively.

   Solution: By Euler, $E \leq 3V - 6$, so in particular the sum of degrees is less than $6V$. We will use this for a contradiction. Suppose there are no 5,6,6 faces. We will count the number of edges which connect vertices of degree 5 to vertices of degree ≥ 7.

   Let x_i be the number of vertices of degree i for each i. No 5,6,6 implies that each 5-vertex has at most 2 neighbors of degree 6, thus it contributes 3 edges which cross from degree 5 to degree ≥ 7. On the other hand, any vertex of degree d has at most $\lfloor d/2 \rfloor$ neighbors of degree 5 because no two degree-5 guys are adjacent. Thus, double-counting gives:

$$\begin{aligned} 3x_5 &\leq \sum_{d=7} x_d \cdot \left\lfloor \frac{d}{2} \right\rfloor \\ x_5 &\leq \sum_{d=7} x_d \cdot \frac{1}{3} \left\lfloor \frac{d}{2} \right\rfloor . \end{aligned}$$

   Note that for $d \geq 7$, the cumbersome expression satisfies $\lfloor d/2 \rfloor / 3 \geq d - 6$. Adding to the LHS so that it becomes 6 times the number of vertices:

$$\begin{aligned} x_5 &\leq \sum_{d=7} x_d \cdot (d-6) \\ 6x_5 + 6x_6 + \sum_{d=7} 6x_d &\leq 5x_5 + 6x_6 + \sum_{d=7} x_d \cdot d. \end{aligned}$$

   Recognize the LHS as $6V$ and the RHS as sum of degrees, and this contradicts our opening observation.

5. (BAMO 2005/4.) There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove

that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

Solution: The key is that 1000 is even. Reduce to the case when the graph is a spanning tree. (One can do breadth-first-search, for example.) Take a leaf and its parent v, and pave the edge between them. Delete both vertices, and let the connected components of the remainder be $C_1, \ldots, C_k$. Note that since we only deleted 2 vertices, the sum of all remaining component sizes is still even, so the number of odd components is even.

We are going to induct into each component, but for this we need all components to be even. For each odd component, add back a copy of v with an edge in the same place it used to be, so that now it is an even graph (but indeed still smaller than the original graph, by at least 1 vertex). Inductively solve every component, and observe that the artificial v's get odd numbers of paved edges an even number of times, since there was an even number of odd components. So still the final v gets an odd number of paved edges because of the inital paving.

6. (IMO Shortlist 2004/C3.) The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from a complete graph on n vertices (where each pair of vertices are joined by an edge).

   Solution: First we show that we cannot end up with any graph with $\leq n-1$ edges. We are only breaking cycles, so we cannot destroy connectivity. Therefore, any final graph with $\leq n-1$ edges must have exactly $n-1$ edges, and be a tree, hence bipartite. But if we consider the reverse process, observe that if we start from a bipartite graph and complete C_4's, we will stay bipartite, and K_n is not bipartite!

   It remains to find an n-edge graph that we can reach. I think one such graph is a triangle plus a single path leading out of one of the vertices of the triangle. See if you can prove this.