Collinearity and concurrence

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1 Warm-up

1. Let $I$ be the incenter of $\triangle ABC$. Let $A'$ be the midpoint of the arc $BC$ of the circumcircle of $\triangle ABC$ which does not contain $A$. Prove that the lines $IA'$, $BC$, and the angle bisector of $\angle BAC$ are concurrent. Hint: you shouldn’t need the Big Point Theorem\footnote{A classic act of desparation in Team Contest presentations.} for this one!

Solution: Two of these lines are the angle bisector of $\angle A$, and of course that intersects with side $BC$.

2 Tools

2.1 Ceva and friends

Ceva. Let $ABC$ be a triangle, with $A' \in BC$, $B' \in CA$, and $C' \in AB$. Then $AA'$, $BB'$, and $CC'$ concur if and only if:

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$ 

Trig Ceva. Let $ABC$ be a triangle, with $A' \in BC$, $B' \in CA$, and $C' \in AB$. Then $AA'$, $BB'$, and $CC'$ concur if and only if:

$$\frac{\sin \angle CA'\!A}{\sin \angle A'\!AB} \cdot \frac{\sin \angle A'B'B}{\sin \angle B'B'C} \cdot \frac{\sin \angle B'C'C}{\sin \angle C'C'A} = 1.$$ 

Menelaus. Let $ABC$ be a triangle, and let $D$, $E$, and $F$ line on the extended lines $BC$, $CA$, and $AB$. Then $D$, $E$, and $F$ are collinear if and only if:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1.$$ 

Now try these problems.

1. (Gergonne point) Let $ABC$ be a triangle, and let its incircle intersect sides $BC$, $CA$, and $AB$ at $A'$, $B'$, $C'$ respectively. Prove that $AA'$, $BB'$, $CC'$ are concurrent.

Solution: Ceva. Since incircle, we have $BA' = CA'$, etc., so Ceva cancels trivially.

2. (Isogonal conjugate of Gergonne point) Let $ABC$ be a triangle, and let $D$, $E$, $F$ be the feet of the altitudes from $A$, $B$, $C$. Construct the incircles of triangles $AEF$, $BDF$, and $CDE$; let the points of tangency with $DE$, $EF$, and $FD$ be $C''$, $A''$, and $B''$, respectively. Prove that $AA''$, $BB''$, $CC''$ concur.

Solution: Trig ceva. Easy to check that triangles $AEF$ and $ABC$ are similar, because, for example, $BFEC$ is cyclic so $\angle ABC = \angle AEF$. Therefore, the line $AA''$ in this problem is the reflection across
the angle bisector of the $AA'$ of the previous problem. So, for example, $\angle CAA'' = \angle A'AB$ and $\angle A''AB = \angle CAA'$.

In particular, since we knew that the previous problem’s $AA'$, $BB'$, and $CC'$ are concurrent, Trig Ceva gives

$$\frac{\sin \angle CAA'}{\sin \angle A'AB} \cdot \frac{\sin \angle A''AB}{\sin \angle B'BC} \cdot \frac{\sin \angle BCC'}{\sin \angle C'CA} = 1.$$ 

Now each ratio flips, because, e.g., $\frac{\sin \angle CAA''}{\sin \angle A'AB} = \frac{\sin \angle A'AB}{\sin \angle CAA'}$. So the product is still $1^{-1} = 1$, hence we have concurrence by Trig Ceva again.

### 2.2 The power of Power of a Point

**Definition.** Let $\omega$ be a circle with center $O$ and radius $r$, and let $P$ be a point. The power of $P$ with respect to $\omega$ is defined to be the difference of squared lengths $OP^2 - r^2$. If $\omega'$ is another circle, then the locus of points with equal power with respect to both $\omega$ and $\omega'$ is called their radical axis.

Use the following exercises to familiarize yourself with these concepts.

1. Let $\omega$ be a circle with center $O$, and let $P$ be a point. Let $\ell$ be a line through $P$ which intersects $O$ at the points $A$ and $B$. Prove that the power of $P$ with respect to $\omega$ is equal to the (signed) product of lengths $PA \cdot PB$.

**Solution:** Classical.

2. Show that the radical axis of two circles is always a line.

**Solution:** You can even use coordinates! Put both circles on $x$-axis, with centers $(x_i, 0)$. Let their radii be $r_i$. Locus is points of the form $(x, y)$ with $(x - x_1)^2 + y^2 - r_1^2 = (x - x_2)^2 + y^2 - r_2^2$. But $y^2$ cancels, and only $x$ remains, so it is a vertical line at the solution $x$.

3. Let $\omega_1$ and $\omega_2$ be two circles intersecting at the points $A$ and $B$. Show that their radical axis is precisely the line $AB$.

**Solution:** Clearly, points $A$ and $B$ have equal power (both zero) with respect to the circles. From previous problem, we know that locus is a line, and two points determine that line.

The above exercises make the following theorem useful.

**Theorem.** (Radical Axis) Let $\omega_1$, $\omega_2$, and $\omega_3$ be three circles. Then their (3) pairwise radical axes are concurrent (or are parallel).

**Proof.** Obvious from transitivity and the above definition of radical axis. □

Now try these problems.

1. (Russia 1997/15) The circles $S_1$ and $S_2$ intersect at $M$ and $N$. Show that if vertices $A$ and $C$ of a rectangle $ABCD$ lie on $S_1$ while vertices $B$ and $D$ lie on $S_2$, then the intersection of the diagonals of the rectangle lies on the line $MN$.

**Solution:** The lines are the radical axes of $S_1$, $S_2$, and the circumcircle of $ABCD$.

2. (USAMO 1997/2) Let $ABC$ be a triangle, and draw isosceles triangles $BCD, CAE, ABF$ externally to $ABC$, with $BC, CA, AB$ as their respective bases. Prove that the lines through $A, B, C$ perpendicular to the lines $EF, FD, DE$, respectively, are concurrent.

**Solution:** Use the three circles: (1) centered at $D$ with radius $DB$, (2) centered at $E$ with radius $EC$, and (3) centered at $F$ with radius $FA$.  

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2.3 Pascal and company

Pappus. Let \( \ell_1 \) and \( \ell_2 \) be lines, let \( A, C, E \in \ell_1 \), and let \( B, D, F \in \ell_2 \). Then \( AB \cap DE \), \( BC \cap EF \), and \( CD \cap FA \) are collinear.

Pascal. Let \( \omega \) be a conic section, and let \( A, B, C, D, E, F \in \omega \). Then \( AB \cap DE \), \( BC \cap EF \), and \( CD \cap FA \) are collinear.

Brianchon. Let the conic \( \omega \) be inscribed in hexagon \( ABCDEF \). Then the diagonals \( AD \), \( BE \), and \( CF \) are concurrent.

Remark. Typically, the only “conics” we need to consider are circles. Also, we can apply to degenerate cases where some of the points coalesce. For example, if we use \( A = B \), then the line \( AB \) should be interpreted as the tangent at \( A \).

Now try these problems.

1. (Half of Bulgaria 1997/10) Let \( ABCD \) be a convex quadrilateral such that \( \angle DAB = \angle ABC = \angle BCD \). Let \( G \) and \( O \) denote the centroid and circumcenter of the triangle \( ABC \). Prove that \( G, O, D \) are collinear. Hint: Construct the following points:
   - \( M \) = midpoint of \( AB \)
   - \( N \) = midpoint of \( BC \)
   - \( E = AB \cap CD \)
   - \( F = DA \cap BC \).

   Solution: Direct application of Pappus to the hexagon \( MCENAF \). Recognize the intersection points as \( G, O, \) and \( D \).

2. (From Kiran Kedlaya’s Geometry Unbound) Let \( ABCD \) be a quadrilateral whose sides \( AB, BC, CD, \) and \( DA \) are tangent to a single circle at the points \( M, N, P, Q \), respectively. Prove that the lines \( AC \), \( BD \), \( MP \), and \( NQ \) are concurrent.

   Solution: Brianchon on \( BNCDQA \) gives concurrence of \( BD, NQ, CA \), and do again on \( AMBCPD \) to get the rest (use transitivity).

3. (Part of MOP 1995/?, also from Kiran) With the same notation as above, let \( BQ \) and \( BP \) intersect the circle at \( E \) and \( F \), respectively. Show that \( B, MP \cap NQ \), and \( ME \cap NF \) are collinear.

   Solution: Pascal on \( EMPFNQ \).

2.4 Shifting targets

Sometimes it is useful to turn a collinearity problem into a concurrence problem, or even to show that different collections of lines/points are concurrent/collinear.

Identification. Three lines \( AB, CD, \) and \( EF \) are concurrent if and only if the points \( A, B, \) and \( CD \cap EF \) are collinear.

Desargues. Two triangles are perspective from a point if and only if they are perspective from a line. Two triangles \( ABC \) and \( DEF \) are perspective from a point when \( AD, BE, \) and \( CF \) are concurrent. Two triangles \( ABC \) and \( DEF \) are perspective from a line when \( AB \cap DE, BC \cap EF, \) and \( CA \cap FD \) are collinear.

False transitivity. If three points are pairwise collinear, that is not enough to ensure that they are collectively collinear, and similarly for lines/concurrence.
**True transitivity.** If distinct points $A, B, C$ and $B, C, D$ are collinear, then all four points are collinear, and similarly for lines/concurrence.

Now try these problems.

1. (Full Bulgaria 1997/10) Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let $H$ and $O$ denote the orthocenter and circumcenter of the triangle $ABC$. Prove that $D, O, H$ are collinear.

**Solution:** In the previous section, we showed that $G, O, D$ were collinear, where $G$ was the centroid of $ABC$. But $G, H, O$ are collinear because they are on the Euler Line of $ABC$, so we are done by transitivity.

2. (Full MOP 1995/?) Let $ABCD$ be a quadrilateral whose sides $AB$, $BC$, $CD$, and $DA$ are tangent to a single circle at the points $M, N, P, Q$, respectively. Let $BQ$ and $BP$ intersect the circle at $E$ and $F$, respectively. Prove that $ME$, $NF$, and $BD$ are concurrent.

**Solution:** Combine previous section’s problems. We know from one of them that $B, MP \cap NQ$, and $D$ are collinear. From the other, we know that $B, MP \cap NQ$, and $ME \cap NF$ are collinear. Identification/transitivity solves the problem.

3 Problems

1. (Zeitz 1996) Let $ABCDEF$ be a convex cyclic hexagon. Prove that $AD, BE, CF$ are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

**Solution:** Trig Ceva

2. (China 1996/1) Let $H$ be the orthocenter of acute triangle $ABC$. The tangents from $A$ to the circle with diameter $BC$ touch the circle at $P$ and $Q$, respectively. Let $BQ$ and $BP$ intersect the circle at $E$ and $F$, respectively. Prove that $M, N, P, Q$ are collinear.

**Solution:** Let $A'$ be the foot of the altitude from $A$, and let $C'$ be the foot of the altitude from $C$. Then $H = AA' \cap CC'$. Let $\omega$ be the circle with diameter $BC$. Construct the circle $\omega'$ with diameter $AO$. The intersection of these two circles is precisely $P, Q$, since $\angle APO = 90^\circ = \angle APO$. So we need to show that $H$ is on the radical axis, i.e., that $H$ has equal power wrt the two circles. Power of $H$ wrt $\omega$ is $CH \cdot HC'$, and power wrt $\omega'$ is $AH \cdot HA'$ since $\angle AA'O = 90^\circ \Rightarrow A' \in \omega'$. But it is a well-known fact that $AH \cdot HA' = CH \cdot HC'$ for any triangle, which can be verified by observing that $ACA'C'$ is cyclic.

3. (Turkey 1996/2) In a parallelogram $ABCD$ with $\angle A < 90^\circ$, the circle with diameter $AC$ meets the lines $CB$ and $CD$ again at $E$ and $F$, respectively, and the tangent to this circle at $A$ meets $BD$ at $P$. Show that $P, F, E$ are collinear.

**Solution:** Use Menelaus. Need to show:

$$\frac{CE}{EB} \cdot \frac{BP}{PD} \cdot \frac{DF}{FC} = -1.$$

Actually, the configuration is already OK, so suffices to consider only unsigned lengths. Construct the point $X = PA \cap CE$. By similar triangles, $BP/DP = BX/DA$. Also by similar triangles, $DF/BE = DA/BA$. So it suffices to show that $CE/FC \cdot BX/BA = 1$, i.e., that $\triangle ABX \sim \triangle ECF$. We already have $\angle B = \angle C$, and we can see that $\angle X = \angle F$ by observing that $\angle X = \frac{1}{2}(AC - AE)$ and $\angle F = \frac{1}{2}EC$, where $AC, AE, EC$ stand for the measures of those arcs in radians. But this is immediate because $AC$ is a diameter.
4. (St. Petersburg 1996/17) The points $A'$ and $C'$ are chosen on the diagonal $BD$ of a parallelogram $ABCD$ so that $AA' || CC'$. The point $K$ lies on the segment $A'C$, and the line $AK$ meets $CC'$ at $L$. A line parallel to $BC$ is drawn through $K$, and a line parallel to $BD$ is drawn through $C$; these meet at $M$. Prove that $D, M, L$ are collinear.

Solution: Can be done with bare hands.

5. (Korea 1997/8) In an acute triangle $ABC$ with $AB \neq AC$, let $V$ be the intersection of the angle bisector of $A$ with $BC$, and let $D$ be the foot of the perpendicular from $A$ to $BC$. If $E$ and $F$ are the intersections of the circumcircle of $AVD$ with $CA$ and $AB$, respectively, show that the lines $AD, BE, CF$ concur.

Solution: Can be done with Ceva.

6. (Bulgaria 1996/2) The circles $k_1$ and $k_2$ with respective centers $O_1$ and $O_2$ are externally tangent at the point $C$, while the circle $k$ with center $O$ is externally tangent to $k_1$ and $k_2$. Let $\ell$ be the common tangent of $k_1$ and $k_2$ at the point $C$ and let $AB$ be the diameter of $k$ perpendicular to $\ell$. Assume that $O$ and $A$ lie on the same side of $\ell$. Show that the lines $AO_1, BO_2, \ell$ have a common point.

Solution: Can be done with Ceva.

7. (Russia 1997/13) Given triangle $ABC$, let $A_1, B_1, C_1$ be the midpoints of the broken lines $CAB$, $ABC$, $BCA$, respectively. Let $l_A, l_B, l_C$ be the respective lines through $A_1, B_1, C_1$ parallel to the angle bisectors of $A, B, C$. Show that $l_A, l_B, l_C$ are concurrent.

Solution: Key observation: $l_A$ passes through the midpoint of $AC$. Since it is parallel to bisector of $\angle A$, and medial triangle is homothety of ratio $-1/2$ of original triangle, the lines $l_A$, etc. concur at the incenter of the medial triangle.

Proof of key observation: construct $B'$ by extending $CA$ beyond $A$ such that $AB' = AB$. Also construct $C'$ by extending $BA$ beyond $A$ such that $AC' = AC$. Then $l_A$ is the line through the midpoints of $BC'$ and $B'C$. This is the midline of quadrilateral $BB'C'C$ parallel to $BB'$, so it hits $BC$ the midpoint of $BC$.

8. (China 1997/4) Let $ABCD$ be a cyclic quadrilateral. The lines $AB$ and $CD$ meet at $P$, and the lines $AD$ and $BC$ meet at $Q$. Let $E$ and $F$ be the points where the tangents from $Q$ meet the circumcircle of $ABCD$. Prove that points $P, E, F$ are collinear.

Solution: Uses Polar Map

4 Harder problems

1. (MOP 1998/2/3a) Let $ABC$ be a triangle, and let $A', B', C'$ be the midpoints of the arcs $BC, CA, AB$, respectively, of the circumcircle of $ABC$. The line $A'B'$ meets $BC$ and $AC$ at $S$ and $T$. $B'C'$ meets $AC$ and $AB$ at $F$ and $P$, and $C'A'$ meets $AB$ and $BC$ at $Q$ and $R$. Prove that the segments $PS, QT, FR$ concur.

Solution: They pass through the incenter of $ABC$, prove with Pascal on $AA'C'B'BC$. See MOP98/2/3a.

2. (MOP 1998/4/5) Let $A_1A_2A_3$ be a nonisosceles triangle with incenter $I$. For $i = 1, 2, 3$, let $C_i$ be the smaller circle through $I$ tangent to $A_iA_{i+1}$ and $A_iA_{i+2}$ (indices being taken mod 3) and let $B_i$ be the second intersection of $C_{i+1}$ and $C_{i+2}$. Prove that the circumcenters of the triangles $A_1B_1I$, $A_2B_2I$, and $A_3B_3I$ are collinear.

Solution: MOP98/4/5: Desargues
3. (MOP 1998/2/3) Let $ABC$ be a triangle, and let $A', B', C'$ be the midpoints of the arcs $BC, CA, AB$, respectively, of the circumcircle of $ABC$. The line $A'B'$ meets $BC$ and $AC$ at $S$ and $T$. $B'C'$ meets $AC$ and $AB$ at $F$ and $P$, and $C'A'$ meets $AB$ and $BC$ at $Q$ and $R$. Prove that the segments $PS, QT, FR$ concur.

Solution: They pass through the incenter of $ABC$, prove with Pascal on $AA'C'B'BC$. See MOP98/2/3a.

4. (MOP 1998/12/3) Let $\omega_1$ and $\omega_2$ be two circles of the same radius, intersecting at $A$ and $B$. Let $O$ be the midpoint of $AB$. Let $CD$ be a chord of $\omega_1$ passing through $O$, and let the segment $CD$ meet $\omega_2$ at $P$. Let $EF$ be a chord of $\omega_2$ passing through $O$, and let the segment $EF$ meet $\omega_1$ at $Q$. Prove that $AB, CQ, EP$ are concurrent.

Solution: MOP98/12/3

5      Impossible problems

- Find (in the plane) a collection of $m$ distinct lines and $n$ distinct points, such that the number of incidences between the lines and points is $> 4(m^{2/3}n^{2/3} + m + n)$. Formally, an incidence is defined as an ordered pair $(\ell, P)$, where $\ell$ is one of the lines and $P$ is one of the points. (This is known to be impossible by the famous Szemerédi-Trotter theorem.)

Solution: The constant of 4 can be obtained via the crossing-lemma argument in the Probabilistic Lens after Chapter 15 in *The Probabilistic Method*, by Alon and Spencer.