B-III. Functional Equations

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Warm-Ups

1. (Russia 2000/9) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(x+y)+f(y+z)+f(z+x) \geq 3f(x+2y+3z)$ for all x, y, z.

   Solution: Answer: f constant. Solution:

   Put $x = a, y = z = 0$, then $2f(a) + f(0) \geq 3f(a)$, so $f(0) \geq f(a)$. Put $x = a/2, y = a/2, z = -a/2$. Then $f(a) + f(0) + f(0) \geq 3f(0)$, so $f(a) \geq f(0)$. Hence $f(a) = f(0)$ for all a. But any constant function obviously satisfies the given relation.

2. (MOP97/2/1) Let f be a real-valued function which satisfies

   (a) for all real x, y, $f(x + y) + f(x - y) = 2f(x)f(y)$.

   (b) there exists a real number x_0 such that $f(x_0) = -1$.

   Prove that f is periodic.

   Solution: Swapping x and y yields that function is even. Yet plugging in $x = y = 0$ we get $f(0) = 0$ or 1. If it is 0, then plugging in $y = 0$ yields $f \equiv 0$, done.

   Otherwise, $f(0) = 1$, and plug in $x = y = x_0/2$ to get $f(x_0)+1 = 2f(x_0/2)^2$, implying that $f(x_0/2) = 0$. Now plugging in $y = x_0/2$, we get that $f(x + x_0/2) = -f(x - x_0/2)$, so function inverts sign every x_0. Hence periodic with period $2x_0$.

3. (Balkan 1987/1) f is a real valued function on the reals satisfying (1) $f(0) = 1/2$, (2) for some real a we have $f(x + y) = f(x)f(a - y) + f(y)f(a - x)$ for all x, y. Prove that f is constant.

   Solution: Put $x = y = 0$. We get $f(0) = 2f(0)f(a)$, so $f(a) = 1/2$. Put $y = 0$, we get $f(x) = f(x)f(a) + f(0)f(a - x)$, so $f(x) = f(a - x)$. Put $y = a - x$, we get $f(a) = f(x)^2 + f(a - x)^2$, so $f(x) = 1/2$ or $-1/2$.

   Now take any x. We have $f(x/2) = 1/2$ or $-1/2$ and $f(a-x/2) = f(x/2)$. Hence $f(x) = f(x/2+x/2) = 2f(x)f(a - x/2) = 1/2$.

Problems

1. (IMO 2002/5) Find all real-valued functions f on the reals such that $[f(x) + f(y)][f(u) + f(v)] = f(xu - yv) + f(xv + yu)$ for all x, y, u, v.

   Solution: Plug in all 0; then $f(0) = 0$ or $1/2$. If $1/2$, then plug in $x = y = 0$ and get $f(u)+f(v) = 1$ implying that constant at $1/2$. Now suppose $f(0) = 0$.

   Plug in $x = v = 0$. Then $f(y)f(u) = f(yu)$. So $f(1)^2 = f(1)$ implying that $f(1) = 0$ or 1. If 0, then multiplicativity implies that constant at 0. Else:

   Plug in $x = y = 1$. Now $2[f(u) + f(v)] = f(u - v) + f(u + v)$. Using $u = 0$, $v = 1$, get $f(-1) = 1$. Multiplicativity implies f is even.

Plug in $x = y, u = v$. Then $4f(x)f(u) = f(2xu)$. Multiplicativity implies that $f(2) = 4$. More multiplicativity gives that $f(x) = x^2$ for all powers of 2. Inductively using $2[f(u)+f(v)] = f(u-v)+f(u+v)$, get that $f(z) = z^2$ for all integers. Reverse multiplicativity implies that $f(q) = q^2$ for all rationals.

Multiplicativity implies $f(x^2) = f(x)^2$ so $f \geq 0$. Yet plug in $x = v, y = u$ and get $f(x^2 + y^2) = [f(x) + f(y)]^2 \geq f(x)^2$ by nonnegativity, so increasing function.

2. (IMO 1999/6) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ for all $x, y \in \mathbb{R}$.

   Solution: Let $c = f(0)$ and A be the image $f(\mathbb{R})$. If a is in A, then it is straightforward to find $f(a)$: putting $a = f(y)$ and $x = a$, we get $f(a - a) = f(a) + a^2 + f(a) - 1$, so $f(a) = (1 + c)/2 - a^2/2$ (*).

   The next step is to show that $A - A = \mathbb{R}$. Note first that c cannot be zero, for if it were, then putting $y = 0$, we get: $f(x - c) = f(c) + xc + f(x) - 1$ (**) and hence $f(0) = f(c) = 1$. Contradiction. But (**) also shows that $f(x - c) - f(x) = xc + (f(c) - 1)$. Here x is free to vary over $\mathbb{R}$, so $xc + (f(c) - 1)$ can take any value in $\mathbb{R}$.

   Thus given any x in $\mathbb{R}$, we may find $a, b \in A$ such that $x = a - b$. Hence $f(x) = f(a - b) = f(b) + ab + f(a) - 1$. So, using (*): $f(x) = c - b^2/2 + ab - a^2/2 = c - x^2/2$.

   In particular, this is true for $x \in A$. Comparing with (*) we deduce that $c = 1$. So for all $x \in \mathbb{R}$ we must have $f(x) = 1 - x^2/2$. Finally, it is easy to check that this satisfies the original relation and hence is the unique solution.

3. Let $f(x)$ be a continuous function with $f(0) = 1$. Suppose that for every $n \in \mathbb{Z}^+$ and any $t \in \mathbb{R}$:

$$(f(t))^n = f\left(\sqrt{n}t\right).$$

   Prove that there exists a constant c such that on $\mathbb{R}^+$, $f(t) = e^{ct^2}$.

   Solution: Suppose there exists $t > 0$ such that $f(t) \leq 0$. Then there exists a minimal $s_0 > 0$ such that $f(s_0) = 0$. But then $f(s_0) = f(s_0/\sqrt{2})^2$, contradicting minimality. Same holds for $t < 0$.

   Therefore, this function exists: $L(t) = \log f(t)$. But then for any $m, n \in \mathbb{Z}^+$, we have $L((n/m)t) = (n/m)^2 L(t)$. Continuity tells us that $L(r) = r^2 L(1)$ for any $r \in \mathbb{R}^+$.

4. (IMO 1996/3) Let S be the set of non-negative integers. Find all functions $f : S \to S$ such that $f(m + f(n)) = f(f(m)) + f(n)$ for all m, n.

   Solution: Setting $m = n = 0$, the given relation becomes: $f(f(0)) = f(f(0)) + f(0)$. Hence $f(0) = 0$. Hence also $f(f(0)) = 0$. Setting $m = 0$, now gives $f(f(n)) = f(n)$, so we may write the original relation as $f(m + f(n)) = f(m) + f(n)$.

   So $f(n)$ is a fixed point. Let k be the smallest non-zero fixed point. If k does not exist, then $f(n)$ is zero for all n, which is a possible solution. If k does exist, then an easy induction shows that $f(qk) = qk$ for all non-negative integers q. Now if n is another fixed point, write $n = kq + r$, with $0 \leq r < k$. Then $f(n) = f(r + f(kq)) = f(r) + f(kq) = kq + f(r)$. Hence $f(r) = r$, so r must be zero. Hence the fixed points are precisely the multiples of k.

   But $f(n)$ is a fixed point for any n, so $f(n)$ is a multiple of k for any n. Let us take $n_1, n_2, \ldots, n_{k-1}$ to be arbitrary non-negative integers and set $n_0 = 0$. Then the most general function satisfying the conditions we have established so far is: $f(qk + r) = qk + n_r k$ for $0 \leq r < k$.

   We can check that this satisfies the functional equation. Let $m = ak + r$, $n = bk + s$, with $0 \leq r, s < k$. Then $f(f(m)) = f(m) = ak + n_r k$, and $f(n) = bk + n_s k$, so $f(m + f(n)) = ak + bk + n_r k + n_s k$, and $f(f(m)) + f(n) = ak + bk + n_r k + n_s k$. So this is a solution and hence the most general solution.

5. (IMO 1994/2) Let S be the set of all real numbers greater than -1. Find all functions $f : S \to S$ such that $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y, and $f(x)/x$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

   Solution: Suppose $f(a) = a$. Then putting $x = y = a$ in the relation given, we get $f(b) = b$, where $b = 2a + a^2$. If $-1 < a < 0$, then $-1 < b < a$. But $f(a)/a = f(b)/b$. Contradiction. Similarly, if $a > 0$, then $b > a$, but $f(a)/a = f(b)/b$. Contradiction. So we must have $a = 0$.

   But putting $x = y$ in the relation given we get $f(k) = k$ for $k = x + f(x) + xf(x)$. Hence for any x we have $x + f(x) + xf(x) = 0$ and hence $f(x) = -x/(x + 1)$.

   Finally, it is straightforward to check that $f(x) = -x/(x + 1)$ satisfies the two conditions.

6. (IMO 1992/2) Find all functions f defined on the set of all real numbers with real values, such that $f(x^2 + f(y)) = y + f(x)^2$ for all x, y.

   Solution: The first step is to establish that $f(0) = 0$. Putting $x = y = 0$, and $f(0) = t$, we get $f(t) = t^2$. Also, $f(x^2 + t) = f(x)^2$, and $f(f(x)) = x + t^2$. We now evaluate $f(t^2 + f(1)^2)$ two ways. First, it is $f(f(1)^2 + f(t)) = t + f(f(1))^2 = t + (1 + t^2)^2 = 1 + t + 2t^2 + t^4$. Second, it is $f(t^2 + f(1 + t)) = 1 + t + f(t)^2 = 1 + t + t^4$. So $t = 0$, as required.

   It follows immediately that $f(f(x)) = x$, and $f(x^2) = f(x)^2$. Given any y, let $z = f(y)$. Then $y = f(z)$, so $f(x^2 + y) = z + f(x)^2 = f(y) + f(x)^2$. Now given any positive x, take z so that $x = z^2$. Then $f(x + y) = f(z^2 + y) = f(y) + f(z)^2 = f(y) + f(z^2) = f(x) + f(y)$. Putting $y = -x$, we get $0 = f(0) = f(x + -x) = f(x) + f(-x)$. Hence $f(-x) = -f(x)$. It follows that $f(x + y) = f(x) + f(y)$ and $f(x - y) = f(x) - f(y)$ hold for all x, y.

   Take any x. Let $f(x) = y$. If $y > x$, then let $z = y - x$. $f(z) = f(y - x) = f(y) - f(x) = x - y = -z$. If $y < x$, then let $z = x - y$ and $f(z) = f(x - y) = f(x) - f(y) = y - x$. In either case we get some $z > 0$ with $f(z) = -z < 0$. But now take w so that $w^2 = z$, then $f(z) = f(w^2) = f(w)^2 \geq 0$. Contradiction. So we must have $f(x) = x$.

7. (Balkan 2000/1) Find all real-valued functions on the reals which satisfy $f(xf(x) + f(y)) = f(x)^2 + y$ for all x, y.

   Solution: Answer: (1) $f(x) = x$ for all x; (2) $f(x) = -x$ for all x.

   Put $x = 0$, then $f(f(y)) = f(0)^2 + y$. Put $y = -f(0)^2$ and $k = f(y)$. Then $f(k) = 0$. Now put $x = y = k$. Then $f(0) = 0 + k$, so $k = f(0)$. Put $y = k, x = 0$, then $f(0) = f(0)^2 + k$, so $k = 0$. Hence $f(0) = 0$.

   Put $x = 0$, $f(f(y)) = y$ (*). Put $y = 0$, $f(xf(x)) = f(x)^2$ (**). Put $x = f(z)$ in (**), then using $f(z) = x$, we have $f(zf(z)) = z^2$. Hence $z^2 = f(z)^2$ for all z (***). In particular, $f(1) = 1$ or -1. Suppose $f(1) = 1$. Then putting $x = 1$ in the original relation we get $f(1 + f(y)) = 1 + y$. Hence $(1 + f(y))^2 = (1 + y)^2$. Hence $f(y) = y$ for all y.

   Similarly if $f(1) = -1$, then putting $x = 1$ in the original relation we get $f(-1 + f(y)) = 1 + y$. Hence $(-1 + f(y))^2 = (1 + y)^2$, so $f(y) = -y$ for all y.

   Finally, it is easy to check that $f(x) = x$ does indeed satisfy the original relation, as does $f(x) = -x$.

8. (IMO 1990/1) Construct a function from the set of positive rational numbers into itself such that $f(xf(y)) = f(x)/y$ for all x, y.

   Solution: We show first that $f(1) = 1$. Taking $x = y = 1$, we have $f(f(1)) = f(1)$. Hence $f(1) = f(f(1)) = f(1f(f(1))) = f(1)/f(1) = 1$.

   Next we show that $f(xy) = f(x)f(y)$. For any y we have $1 = f(1) = f(1/f(y)f(y)) = f(1/f(y))/y$, so if $z = 1/f(y)$ then $f(z) = y$. Hence $f(xy) = f(xf(z)) = f(x)/z = f(x)f(y)$.

   Finally, $f(f(x)) = f(1f(x)) = f(1)/x = 1/x$.

   We are not required to find all functions, just one. So divide the primes into two infinite sets S = $\{p_1, p_2, \ldots\}$ and $T = \{q_1, q_2, \ldots\}$. Define $f(p_n) = q_n$, and $f(q_n) = 1/p_n$. We extend this definition to all rationals using $f(xy) = f(x)f(y)$: $f(p_{i_1}p_{i_2} \cdots q_{j_1}q_{j_2} \cdots /(p_{k_1} \cdots q_{m_1} \cdots)) = p_{m_1} \cdots q_{i_1} \cdots /(p_{j_1} \cdots q_{k_1} \cdots)$. It is now trivial to verify that $f(xf(y)) = f(x)/y$.

9. (IMO Shortlist 1995/A5) Does there exist a real-valued function f on the reals such that $f(x)$ is bounded, $f(1) = 1$ and $f(x + 1/x^2) = f(x) + f(1/x)^2$ for all non-zero x?

Solution: Answer: no.

Suppose there is such a function. Let c be the least upper bound of the set of values $f(x)$. We have $f(2) = f(1 + 1/12) = f(1) + f(1/1)^2 = 2$. So $c \geq 2$. But definition we can find y such that $f(y) > c - 1/4$. So $c \geq f(y + 1/y^2) = f(y) + f(1/y)^2 > c - 1/4 + f(1/y)^2$. So $f(1/y)^2 < 1/4$ and hence $f(1/y) > -1/2$.

We also have $c \geq f(1/y+y^2) = f(1/y)+f(y)^2 > -1/2+(c-1/4)^2 = c^2-c/2-7/16$. So $c^2-3c/2-7/16 < 0$, or $(c - 3/4)^2 < 1$. But $c \geq 2$, so that is false. Contradiction. So there cannot be any such function.