1 Warm-Ups

1. Consider the cubic equation \( ax^3 + bx^2 + cx + d = 0 \). The roots are

\[
x = \frac{-b^3 + \frac{bc}{6a^2} - \frac{d}{2a}}{27a^3} + \frac{-b^3 + \frac{bc}{6a^2} - \frac{d}{2a}}{27a^3} + \frac{c}{3a} - \frac{b^2}{9a^2}
\]

Prove that no such general formula exists for a quintic equation.

2 Theory

Thanks to Elgin Johnston (1997) for these theorems.

**Rational Root Theorem** Let \( p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \) be a polynomial with integer coefficients. Then any rational solution \( \frac{r}{s} \) (expressed in lowest terms) must have \( r|a_0 \) and \( s|a_n \).

**Descartes’s Rule of Signs** Let \( p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \) be a polynomial with real coefficients. Then the number of positive roots is equal to \( N - 2k \), where \( N \) is the number of sign changes in the coefficient list (ignoring zeros), and \( k \) is some nonnegative integer.

**Eisenstein’s Irreducibility Criterion** Let \( p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \) be a polynomial with integer coefficients and let \( q \) be a prime. If \( q \) is a factor of each of \( a_{n-1}, a_{n-2}, \ldots, a_0 \), but \( q \) is not a factor of \( a_n \), and \( q^2 \) is not a factor of \( a_0 \), then \( p(x) \) is irreducible over the rationals.

**Einstein’s Theory of Relativity** Unfortunately, this topic is beyond the scope of this program.

**Gauss’s Theorem** If \( p(x) \) has integer coefficients and \( p(x) \) can be factored over the rationals, then \( p(x) \) can be factored over the integers.

**Lagrange Interpolation** Suppose we want a degree-\( n \) polynomial that passes through a set of \( n+1 \) points: \( \{(x_i, y_i)\}_{i=0}^n \). Then the polynomial is:

\[
p(x) = \sum_{i=0}^{n} \frac{y_i}{\text{normalization}} (x - x_0)(x - x_1) \cdots (x - x_n),
\]

where the \( i \)-th “normalization” factor is the product of all the terms \( (x_i - x_j) \) that have \( j \neq i \).
3 Problems

Thanks to Elgin Johnston (1997) for most of these problems.

1. (Crux Math., June/July 1978) Show that \( n^4 - 20n^2 + 4 \) is composite when \( n \) is any integer.

Solution: Factor as difference of two squares. Prove that neither factor can be \( \pm 1 \).

2. (St. Petersburg City Math Olympiad 1998/14) Find all polynomials \( P(x,y) \) in two variables such that for any \( x \) and \( y \), \( P(x+y, y-x) = P(x, y) \).

Solution: Clearly constant polynomials work. Also, \( P(x,y) = P(x+y, y-x) = P(2y, -2x) = P(16x, 16y) \). Suppose we have a nontrivial polynomial. Then on the unit circle, it is bounded because we can just look at the fixed coefficients. Yet along each ray \( y = tx \), we get a polynomial whose translate has infinitely many zeros, so it must be constant. Hence \( P \) is constant along all rays, implying that \( P \) is bounded by its max on the unit circle, hence bounded everywhere. Now suppose maximum degree of \( y \) is \( N \). Study the polynomial \( P(z^{N+1}, z) \). The leading coeff of this is equal to the leading coeff of \( P(x,y) \) when sorted with respect to \( x \) as more important. Since the \( z \)-poly is also bounded everywhere, it too must be constant, implying that the leading term is a constant.

3. (Putnam, May 1977) Determine all solutions of the system

\[
\begin{align*}
x + y + z &= w \\
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w}.
\end{align*}
\]

Solution: Given solutions \( x, y, z \), construct 3-degree polynomial \( P(t) = (t-x)(t-y)(t-z) \). Then \( P(t) = t^3 - wt^2 + At - Aw = (t^3 + A)(t - w) \). In particular, roots are \( w \) and a pair of opposites.

4. (Crux Math., April 1979) Determine the triples of integers \( (x, y, z) \) satisfying the equation

\[ x^3 + y^3 + z^3 = (x + y + z)^3. \]

Solution: Move \( z^3 \) to RHS and factor as \( x^3 \pm y^3 \). We get \( (x + y) = 0 \) or \((y + z)(z + x) = 0 \). So two are opposites.

5. (USSR Olympiad) Prove that the fraction \((n^3 + 2n)/(n^4 + 3n^2 + 1)\) is in lowest terms for every positive integer \( n \).

Solution: Use Euclidean algorithm for GCD. \((n^3 + 2n)n = n^4 + 2n^2 \), so difference to denominator is \( n^5 + 1 \). Yet that’s relatively prime to \( n(n^2 + 2) \).

6. (Po, 2004) Prove that \( x^4 - x^3 - 3x^2 + 5x + 1 \) is irreducible.

Solution: Eisenstein with substitution \( x \mapsto x + 1 \).

7. (Canadian Olympiad, 1970) Let \( P(x) \) be a polynomial with integral coefficients. Suppose there exist four distinct integers \( a, b, c, d \) with \( P(a) = P(b) = P(c) = P(d) = 5 \). Prove that there is no integer \( k \) with \( P(k) = 8 \).

Solution: Drop it down to 4 zeros, and check whether one value can be 3. Factor as \( P(x) = (x-a)(x-b)(x-c)(x-d)R(x) \); then substitute \( k \). 3 is prime, but we’ll get at most two \( \pm 1 \) terms from the \((x-a)\) product.

8. (Monthly, October 1962) Prove that every polynomial over the complex numbers has a nonzero polynomial multiple whose exponents are all divisible by \( 10^9 \).

Solution: Factor polynomial as \( a(x-r_1)(x-r_2)\cdots(x-r_n) \). Then the desired polynomial is \( a(x^p - r_1^p)\cdots(x^p - r_n^p) \), where \( P = 10^9 \). Each factor divides the corresponding factor.
9. (Elgin, MOP 1997) For which \( n \) is the polynomial \( 1 + x^2 + x^4 + \cdots + x^{2n-2} \) divisible by the polynomial \( 1 + x + x^2 + \cdots + x^{n-1} \)?

**Solution:** Observe:

\[
\begin{align*}
(x^2 - 1)(1 + x^2 + x^4 + \cdots + x^{2n-2}) &= x^{2n} - 1 \\
(x - 1)(1 + x + x^2 + \cdots + x^n) &= x^n - 1 \\
(x + 1)(1 + x^2 + x^4 + \cdots + x^{2n-2}) &= (x^n + 1)(1 + x + x^2 + \cdots + x^{n-1}).
\end{align*}
\]

So if the quotient is \( Q(x) \), then \( Q(x)(x+1) = x^n + 1 \). This happens iff \(-1\) is a root of \( x^n + 1 \), which is iff \( n \) is odd.

10. (Czech-Slovak Match, 1998/1) A polynomial \( P(x) \) of degree \( n \geq 5 \) with integer coefficients and \( n \) distinct integer roots is given. Find all integer roots of \( P(P(x)) \) given that 0 is a root of \( P(x) \).

**Solution:** Answer: just the roots of \( P(x) \). Proof: write \( P(x) = x(x-r_1)(x-r_2)\cdots(x-r_N) \). Suppose we have another integer root \( r \); then \( r(r-r_1)\cdots(r-r_N) = r_k \) for some \( k \). Since degree is at least 5, this means that we have \( 2r(r-r_k) \) dividing \( r_k \). Simple analysis shows that \( r \) is between 0 and \( r_k \); more analysis shows that we just need to defuse the case of \( 2ab \mid a+b \). Assume \( a \leq b \). Now if \( a = 1 \), only solution is \( b = 1 \), but then we already used \( \pm 1 \) in the factors, so we actually have to have \( 12r(r-r_k) \) dividing \( r_k \), no good. If \( a > 1 \), then \( 2ab > 2b \geq a+b \), contradiction.

11. (Hungarian Olympiad, 1899) Let \( r \) and \( s \) be the roots of

\[ x^2 - (a+d)x + (ad-bc) = 0. \]

Prove that \( r^3 \) and \( s^3 \) are the roots of

\[ y^2 - (a^3 + d^3 + 3abc + 3bcd)y + (ad-bc)^3 = 0. \]

**Hint:** use Linear Algebra.

**Solution:** \( r \) and \( s \) are the eigenvalues of the matrix

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

The \( y \) equation is the characteristic polynomial of the cube of that matrix.

12. (Hungarian Olympiad, 1981) Show that there is only one natural number \( n \) such that \( 2^8 + 2^{11} + 2^n \) is a perfect square.

**Solution:** \( 2^8 + 2^{11} = 48^2 \). So, need to have \( 2^n \) as difference of squares \( N^2 - 48^2 \). Hence \((N+48), (N-48)\) are both powers of 2. Their difference is 96. Difference between two powers of 2 is of the form \( 2^M(2^N - 1) \). Uniquely set to \( 2^7 - 2^5 \).

13. (MOP 97/9/3) Let \( S = \{s_1, s_2, \ldots, s_n\} \) be a set of \( n \) distinct complex numbers, for some \( n \geq 9 \), exactly \( n-3 \) of which are real. Prove that there are at most two quadratic polynomials \( f(z) \) with complex coefficients such that \( f(S) = S \) (that is, \( f \) permutates the elements of \( S \)).

14. (MOP 97/9/1) Let \( P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \) be a nonzero polynomial with integer coefficients such that \( P(r) = P(s) = 0 \) for some integers \( r \) and \( s \), with \( 0 < r < s \). Prove that \( a_k \leq -s \) for some \( k \).