VIII. Brutal Force 2003

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brutal (adj.)

1. Extremely ruthless or cruel.
2. Crude or unfeeling in manner or speech.
3. Harsh; unrelenting.
4. Disagreeably precise or penetrating.

force (n.)

1. The capacity to do work or cause physical change; energy, strength, or active power.
2. Power made operative against resistance; exertion.
3. A vector quantity that tends to produce an acceleration of a body in the direction of its application.
4. A unit of a nation's military personnel, especially one deployed into combat.

1 Warm-Ups

1. Do 20 pushups on your knuckles.
2. (Russia98/14). A circle S centered at O meets another circle S' at A and B. Let C be a point on the arc of S contained in S'. Let E, D be the second intersections of S' with AC, BC, respectively. Show that $DE \perp OC$.

   Solution: Clearly true for symmetric case; for perturbation, angles flow around as usual.
3. (UK96/3). Let ABC be an acute triangle and O its circumcenter. Let S denote the circle through A, B, O. The lines CA and CB meet S again at P and Q, respectively. Prove that the lines CO and PQ are perpendicular.
4. (Rookie Contests 1998, 1999, 2002 / Po's "Star" Theorem). Given two congruent circles, ω_1 and ω_2. Let them intersect at B and C. Select a point A on ω_1. Let AB and AC intersect ω_2 at A_1 and A_2. Let X be the midpoint of BC. Let A_1X and A_2X intersect ω_1 at P_1 and P_2. Prove that $AP_1 = AP_2$.

   Solution: True for symmetric case; perturb A by θ. Then A_1 and A_2 move by θ (vertical angles), and P_1 and P_2 also move by θ (symmetry through X). Therefore done.

2 (Mostly) Brutalizable Problems

You can't use Brutal Force on everything. To let you practice the art of figuring out whether a problem is brutalizable, here's a mixed list of problems: some yield to Brutal Force, but some do not. Figure out which ones do, and dish out the PWNage!

1. (IMOshortlist90/12). Let ABC be a triangle with angle bisectors AD and BF. The lines AD, BF meet the line through C parallel to AB at E and G respectively, and $FG = DE$. Show that $CA = CB$.

   Solution: See sheet
2. (Steiner-Lehmus). Prove that any triangle with two equal-length angle bisectors is isosceles.
3. (APMO00/3). Let ABC be a triangle. The angle bisector at A meets the side BC at X. The perpendicular to AX at X meets AB at Y. The perpendicular to AB at Y meets the ray AX at R. XY meets the median from A at S. Prove that RS is perpendicular to BC.

   Solution: See sheet

4. (Russia02/3). Let O be the circumcenter of acute triangle ABC with $AB = BC$. Point M lies on segment BO, and point M' is the reflection of M across the midpoint of side AB. Point K is the intersection of lines $M'O$ and AB. Point L lies on side BC such that $\angle CLO = \angle BLM$. Show that O, K, B, L are concyclic.

   Solution: See sheet

5. (Russia98/10). Let ABC be an acute triangle, and let S be the circle that passes through the circumcenter O and vertices B, C. Let OK be a diameter of S, and let D, E be the second intersections of S with AB, AC, respectively. Show that $ADKE$ is a parallelogram.

   Solution: In symmetric case, A, O, K collinear. Therefore, $\angle OKD = \angle OKE = \angle OCE = \angle OCA$ since $OA = OC$. But perturb A by θ; then D and E also move by θ and we are done.

6. (IMO87/2). Let ABC be an acute-angled triangle, where the interior bisector of angle A meets BC at L and meets the circumcircle of ABC again at N. From L perpendiculars are drawn to AB and AC, with feet K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.

   Solution: See sheet

7. (MOP98/4/5). Suppose $A_1A_2A_3$ is a nonisosceles triangle with incenter I. For $i = 1, 2, 3$, let C_i be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (indices being taken mod 3) and let B_i be the second intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, and A_3B_3I are collinear.

   Solution: MOP98/4/5

8. (IMO02/2). Let BC be a diameter of a circle centered at O. A is any point on the circle with $\angle AOC > 60°$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.

   Solution: See sheet

9. (Lemma). Given a triangle ABC, and D on the extension of ray BC past C. Construct an arbitrary line ℓ through D such that it passes through the interior of triangle ABC. Let $E = \ell \cap AB$ and $F = \ell \cap AC$. What is the locus of $BF \cap CE$?

10. (Lemming). Given a triangle ABC, and D on the extension of ray BC past C. Construct an arbitrary line ℓ through D such that it passes through the interior of triangle ABC. Let $E = \ell \cap AB$ and $F = \ell \cap AC$. What is the locus of $BE \cap CF$?

11. (MOP97/2/5). ABC is a triangle and D, E, F are the points where its incircle touches sides BC, CA, AB, respectively. The parallel through E to AB intersects DF in Q, and the parallel through D to AB intersects EF in T. Prove that CF, DE, QT are concurrent.

    Solution: MOP97/2/5

12. (MOP97/10/4). Let ABC be a triangle with $\angle A = 120°$. Let P be a point on the angle bisector AD of $\angle A$, and let E be the intersection of CP with AB and F the intersection of ED with BP. Find the angle $\angle FAD$.

    Solution: See sheet

13. (Russia01/28). Let AC be the longest of the three sides in triangle ABC. Let N be a point on AC. Let the perpendicular bisector of AN intersect line AB at K, and let the perpendicular bisector of CN intersect line BC at M. Prove that the circumcenter of triangle ABC lies on the circumcircle of triangle KBM.

    Solution: See sheet

14. (StP97/19). Let circles S_1, S_2 intersect at A and B. Let Q be a point on S_1. The lines QA and QB meet S_2 at C and D, respectively, while the tangents to S_1 at A and B meet at P. Assume that Q lies outside S_2, and that C and D lie outside S_1. Prove that the line QP goes through the midpoint of CD.

   Solution: Clearly true for symmetric case; perturb Q by θ and observe that the midpoint M orbits a circle ω centered at O_2 (center of S_2) by θ. Furthermore, by limiting argument as Q approaches A or B, we see that the tangents AP and BP are also tangent to ω. Now we have S_1 and ω with common tangents AP and BP; therefore, by homothety we are done.

15. (MOP98/9/5). ABC is a triangle, and let D, E be the second intersections of the circle with diameter BC with the lines AB, AC, respectively. Let F, G be the feet of the perpendiculars from D, E, respectively, to BC, and let $M = DG \cap EF$. Prove that AM is perpendicular to BC.

   Solution: MOP98/9/5

16. (MOP98/IMO3/3). Let circle ω_1, centered at O_1, and circle ω_2, centered at O_2, meet at A and B. Let ℓ be a line through A meeting ω_1 at Y and meeting ω_2 again at Z. Let X be the intersection of the tangent to ω_1 at Y and the tangent to ω_2 at Z. Let ω be teh circumcircle of O_1O_2B, and let Q be the second intersection of ω with BX. Prove that the length of XQ equals the diameter of ω.

   Solution: MOP98/IMO3/3

Hint: brutalizable problems Let you know who they are.