1 Warm-Ups

1. Line up sequentially in height-order, and say “Cauchy.”

2. (Abel Summation) Suppose that we have \((a_k)_1^n\) and \((b_k)_1^n\). Also, suppose that we define \(S_k = \sum_{i=1}^k a_i\).

Then:

\[
\sum_{k=1}^n a_k b_k = S_n b_n + \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}).
\]

2 Problems

1. Calculate the sum \(\sum_{k=1}^n \frac{k}{2^k}\).

Solution: Split it into:

\[
\sum_{k=1}^n \frac{k}{2^k} = \sum_{k=1}^n \sum_{i=k}^n \frac{1}{2^i}
\]

Now use geometric series summation to get \(2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}\).

2. Prove that \(16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17\).

Solution: Divide the sum by 2, and then substitute the denominator with \((\sqrt{k} + \sqrt{k+1})\), with appropriate adjustment for the two directions.

3. Let \((a_k)_1^n\) be a positive sequence. Let \((b_k)_1^n\) be a real sequence (not necessarily positive). Suppose that \(\sum_{i \neq j} a_i b_j = 0\). Prove that \(\sum_{i \neq j} b_i b_j \leq 0\).

Solution: Let \(a = \sum a_k\) and \(b = \sum b_k\). The given tells us that \(ab = \sum a_k b_k\). The result is equivalent to \(b^2 \leq \sum b_k^2\). By Cauchy-Schwarz:

\[
(ab)^2 = (\sum a_k b_k)^2 \leq (\sum a_k^2)(\sum b_k^2) \leq a^2 \sum b_k^2,
\]

since \((a)\) is positive. But then we get precisely that \(b^2 \leq \sum b_k^2\).

4. Let \((a_k)_1^n\) and \((b_k)_1^n\) be two real sequences, and suppose that \((b_k)\) is nonnegative and decreasing. For \(k \in \{1, 2, \ldots, n\}\), define \(S_k = \sum_{i=1}^k a_i\). Let \(M = \max\{S_1, \ldots, S_n\}\) and \(m = \min\{S_1, \ldots, S_n\}\). Prove that

\[
mb_1 \leq \sum_{i=1}^n a_i b_i \leq Mb_1
\]
5. Let \((a_k)\) and \((b_k)\) be two real sequences with \(a\) nonnegative and decreasing. Also suppose that \(\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i\) for all \(k\). Prove that \(\sum_{i=1}^{n} a_i^2 \leq \sum_{i=1}^{n} b_i^2\).

**Solution:** Use Abel sum; get \(\sum a_i^2 \leq S_n a_n + \sum b_k (a_k - a_{k+1}) = b_k a_k\). But by Cauchy-Schwarz:

\[
(\sum b_k a_k)^2 \leq (\sum a_i^2)(\sum b_k^2) \leq (\sum a_k b_k)(\sum b_k^2).
\]

Divide through both sides (it is positive since it is bigger than \(\sum a_i^2\)) and we get that \(\sum b_k a_k \leq b_k^2\) and the result follows by transitivity.

6. (IMO 78) Let \((a_k)\) be a sequence of distinct positive integers. Prove that for any positive integer \(n\):

\[
\sum_{k=1}^{n} a_k^2 \geq \sum_{k=1}^{n} \frac{1}{k^2}.
\]

**Solution:** Rearrangement

7. (USAMO 89) For each positive integer \(n\), let:

\[
S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},
\]

\[
T_n = S_1 + S_2 + \cdots + S_n,
\]

\[
U_n = \frac{T_1}{3} + \frac{T_2}{4} + \cdots + \frac{T_n}{n+1}.
\]

Find integers \(0 < a, b, c, d < 1000000\) for which \(T_{1998} = a S_{1989} - b\) and \(U_{1988} = c S_{1989} - d\).

**Solution:** For the first one, write out the sum in table-form with it on horizontals, and add columns. We will get that \(T_n = (n+1)S_n - (n+1)\). Hence the terms of the \(U_n\) sum are simply \((S_n - 1)\). Plug this back in and use the previous; we get answers of \(a = b = 1989, c = 1990, d = 3978\).

8. Given two real sequences \((a_k)\) and \((b_k)\) with

(a) \(a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\)
(b) \(b_1 \geq a_1\) and \(b_1 b_2 \geq a_1 a_2\) and \(b_1 b_2 \cdots b_n \geq a_1 a_2 \cdots a_n\).

Prove that \(\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} a_i^2\) and determine the condition of equality.

**Solution:** Weighted AM-GM:

\[
\frac{\sum a_i^2}{\sum a_i} \geq \sqrt[n]{\prod \left(\frac{b_i}{a_i}\right)^{a_i}}
\]

Now divide through in the given conditions and take the \((a_i/b_i)\) term by taking the \((b_1 b_2 \cdots b_n)/(a_1 a_2 \cdots a_n) \geq 1\). Since \((a_i)\) is decreasing, we can continue in this way without making any of the powers negative.

9. (USAMO 94) Let \((a_k)\) be a positive sequence satisfying \(\sum_{j=1}^{n} a_j \geq \sqrt{n}\) for all \(n \geq 1\). Prove that for all \(n \geq 1\):

\[
\sum_{j=1}^{n} a_j^2 > \frac{1}{4} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).
\]

**Solution:** Use result from three problems ago; use the \(a_j^2\) for the \(b_k\), and add in \(a_k = 1/(2\sqrt{k})\).
10. Given two real sequences \((a_k)_{k=1}^n\) and \((b_k)_{k=1}^n\), prove that
\[
\sum_{i=1}^n a_i x_i \leq \sum_{i=1}^n b_i x_i \text{ for any } x_1 \leq x_2 \leq \cdots \leq x_n
\]
is equivalent to
\[
\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \text{ and } \sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i, \text{ for } k = 1, 2, \ldots, n-1.
\]

**Solution:** Abel sum; taking \(\Delta_k \equiv 0\), \(x_n = 1\), we get the equality of full sums. For \(x_n = 0\), \(\Delta_k = \delta_k\), we get the rest of it. For the converse, it simply plugs into the Abel sum.

11. (USAMO 1985) \(0 < a_1 \leq a_2 \leq a_3 \leq \ldots\) is an unbounded sequence of integers. Let \(b_n = m\) if \(a_m\) is the first member of the sequence to equal or exceed \(n\). Given that \(a_{19} = 85\), what is the maximum possible value of
\[
a_1 + a_2 + \cdots + a_{19} + b_1 + b_2 + \cdots + b_{85}?
\]

**Solution:** If all \(a_k\) are 85, then we get 1700. But use algorithm to turn any sequence into flat-85: if \(a_k < a_{k+1}\), then can replace \(a_k\) by \(a_k + 1\). This will increase the sum of \(a_i\) by 1, but decrease the sum of \(b_i\) by 1.

12. (Uses Calculus) Find a compact expression for \(\sum_{k=1}^n k x^k\).

3 **Really Hard Zuming Problem**

Let \((a_k)_{k=1}^n\) be a positive sequence. Prove that:
\[
\sum_{k=1}^n \frac{k}{a_1 + a_2 + \cdots + a_k} < 2 \sum_{i=1}^n \frac{1}{a_i}.
\]

Hint: use the following lemma:

1. For intermediate \(k\):
\[
\frac{k}{\sum_{i=1}^k a_i} \leq \frac{4}{k(k+1)^2} \sum_{i=1}^k i^2 / a_i
\]

**Solution:** Cauchy-Schwarz with \((k), (a_k), \text{ and } (k/\sqrt{a_k})\).

**Solution:** Now do this:
\[
\sum_{i=1}^n \frac{k}{\sum_{i=1}^k a_i} \leq \sum_{i=1}^n \frac{4k}{k^2(k+1)^2} \sum_{i=1}^k i^2 / a_i
\]
\[
< 2 \sum_{i=1}^n \frac{2k + 1}{k^2(k+1)^2} \sum_{i=1}^k i^2 / a_i
\]
\[
= 2 \sum_{k=1}^n \sum_{i=1}^n \frac{i^2}{a_i} \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right)
\]
\[
= 2 \sum_{i=1}^n \sum_{k=1}^n \frac{i^2}{a_i} \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right)
\]
\[
\begin{align*}
= & \quad 2 \sum_{i=1}^{n} \frac{i^2}{a_i} \left( \frac{1}{i^2} - \frac{1}{(n+1)^2} \right) \\
< & \quad 2 \sum_{i=1}^{n} \frac{1}{a_i}
\end{align*}
\]