VII. Sequences (from Zuming97/98)

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1 Warm-Ups

1. Line up sequentially in height-order, and say "Cauchy."
2. (Abel Summation) Suppose that we have $(a_k)_1^n$ and $(b_k)_1^n$. Also, suppose that we define $S_k = \sum_{i=1}^k a_i$. Then:

$$\sum_{k=1}^{n} a_k b_k = S_n b_n + \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}).$$

2 Problems

1. Calculate the sum $\sum_{k=1}^n k/(2^k)$.

   Solution: Split it into:

$$\sum_1^n \frac{k}{2^k} = \sum_{k=1}^n \sum_{i=k}^n \frac{1}{2^i}.$$

   Now use geometric series summation to get $2 - 1/(2^{n-1}) - n/(2^n)$.

2. Prove that $16 < \sum_{k=1}^{80} 1/\sqrt{k} < 17$.

   Solution: Divide the sum by 2, and then substitute the denominator with $(\sqrt{k} + \sqrt{k+1})$, with appropriate adjustment for the two directions.

3. Let $(a_k)_1^n$ be a positive sequence. Let $(b_k)_1^n$ be a real sequence (not necessarily positive). Suppose that $\sum_{i \neq j} a_i b_j = 0$. Prove that $\sum_{i \neq j} b_i b_j \leq 0$.

   Solution: Let $a = \sum a_k$ and $b = \sum b_k$. The given tells us that $ab = \sum a_k b_k$. The result is equivalent to $b^2 \leq \sum b_k^2$. By Cauchy-Schwarz:

$$(ab)^2 = (\sum a_k b_k)^2 \leq (\sum a_k^2)(\sum b_k^2) \leq a^2 \sum b_k^2,$$

   since (a) is positive. But then we get precisely that $b^2 \leq \sum b_k^2$.

4. Let $(a_k)_1^n$ and $(b_k)_1^n$ be two real sequences, and suppose that (b_k) is nonnegative and decreasing. For $k \in \{1, 2, \ldots, n\}$, define $S_k = \sum_{i=1}^k a_i$. Let $M = \max\{S_1, \ldots, S_n\}$ and $m = \min\{S_1, \ldots, S_n\}$. Prove that

$$mb_1 \leq \sum_{i=1}^n a_i b_i \leq Mb_1$$

5. Let $(a_k)_1^n$ and $(b_k)_1^n$ be two real sequences with (a) nonnegative and decreasing. Also suppose that $\sum_{i=1}^k a_i \le \sum_{i=1}^k b_i$ for all k. Prove that $\sum_{i=1}^n a_i^2 \le \sum_{i=1}^n b_i^2$.

   Solution: Use Abel sum; get $\sum a_i^2 \le S_{bn} a_n + \sum S_{bk}(a_k - a_{k+1}) = \sum b_k a_k$. But by Cauchy-Schwarz:

$$(\sum b_k a_k)^2 \le (\sum a_k^2)(\sum b_k^2) \le (\sum a_k b_k)(\sum b_k^2).$$

   Divide through both sides (it is positive since it is bigger than $\sum a_k^2$) and we get that $\sum b_k a_k \le \sum b_k^2$ and the result follows by transitivity.

6. (IMO78) Let $(a_k)_1^n$ be a sequence of distinct positive integers. Prove that for any positive integer n:

$$\sum_{k=1}^n \frac{a_k}{k^2} \ge \sum_{k=1}^n \frac{1}{k}.$$

   Solution: Rearrangement

7. (USAMO89) For each positive integer n, let:

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ T_n &= S_1 + S_2 + \cdots + S_n \\ U_n &= \frac{T_1}{2} + \frac{T_2}{3} + \cdots + \frac{T_n}{n+1}. \end{aligned}$$

   Find integers $0 < a, b, c, d < 1000000$ for which $T_{1998} = aS_{1989} - b$ and $U_{1988} = cS_{1989} - d$.

   Solution: For the first one, write out the sum in table-form with it on horizontals, and add columns. We will get that $T_n = (n+1)S_n - (n+1)$. Hence the terms of the U_n sum are simply $(S_n - 1)$. Plug this back in and use the previous; we get answers of $a = b = 1989, c = 1990, d = 3978$.

8. Given two real sequences $(a_k)_1^n$ and $(b_k)_1^n$ with

   (a) $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$

   (b) $b_1 \ge a_1$ and $b_1 b_2 \ge a_1 a_2$ and $\ldots$ and $b_1 b_2 \cdots b_n \ge a_1 a_2 \cdots a_n$.

   Prove that $\sum_{i=1}^n b_i \ge \sum_{i=1}^n a_i$ and determine the condition of equality.

   Solution: Weighted AM-GM:

$$\frac{\sum a_i \frac{b_i}{a_i}}{\sum a_i} \ge \sqrt[\sum a_i]{\prod \left(\frac{b_i}{a_i}\right)^{a_i}}$$

   Now divide through in the given conditions and take the $(b_n/a_n)_n^a$ term by taking the $(b_1 b_2 \cdots b_n)/(a_1 a_2 \cdots a_n) \ge 1$. Since (a) is decreasing, we can continue in this way without making any of the powers negative.

9. (USAMO94) Let $(a_k)_1^n$ be a positive sequence satisfying $\sum_{j=1}^n a_j \ge \sqrt{n}$ for all $n \ge 1$. Prove that for all $n \ge 1$:

$$\sum_{j=1}^n a_j^2 > \frac{1}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

   Solution: Use result from three problems ago; use the a_j for the b_k, and add in $a_k = 1/(2\sqrt{k})$.

10. Given two real sequences $(a_k)_1^n$ and $(b_k)_1^n$, prove that

$$\sum_{i=1}^n a_i x_i \le \sum_{i=1}^n b_i x_i \text{ for any } x_1 \le x_2 \le \cdots \le x_n$$

is equivalent to

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \text{ and } \sum_{i=1}^k a_i \ge \sum_{i=1}^k b_i, \text{ for } k = 1, 2, \ldots, n-1.$$

Solution: Abel sum; taking $\Delta_k \equiv 0$, $x_n = 1$, we get the equality of full sums. For $x_n = 0$, $\Delta_k = \delta_k$, we get the rest of it. For the converse, it simply plugs into the Abel sum.

11. (USAMO85) $0 < a_1 \le a_2 \le a_3 \le \ldots$ is an unbounded sequence of integers. Let $b_n = m$ if a_m is the first member of the sequence to equal or exceed n. Given that $a_{19} = 85$, what is the maximum possible value of $a_1 + a_2 + \cdots + a_{19} + b_1 + b_2 + \cdots + b_{85}$?

Solution: If all a_k are 85, then we get 1700. But use algorithm to turn any sequence into flat-85: if $a_k < a_{k+1}$, then can replace a_k by $a_k + 1$. This will increase the sum of a_i by 1, but decrease the sum of b_i by 1.

12. (Uses Calculus) Find a compact expression for $\sum_{k=1}^n kx^k$.

3 Really Hard Zuming Problem

Let $(a_k)_1^n$ be a positive sequence. Prove that:

$$\sum_{k=1}^n \frac{k}{a_1 + a_2 + \cdots + a_k} < 2 \sum_{i=1}^n \frac{1}{a_i}.$$

Hint: use the following lemma:

1. For intermediate k:

$$\frac{k}{\sum_{i=1}^k a_i} \le \frac{4}{k(k+1)^2} \sum_{i=1}^k \frac{i^2}{a_i}$$

Solution: Cauchy-Schwarz with (k), (a_k), and $(k/\sqrt{a_k})$.

Solution: Now do this:

$$\begin{aligned}
\sum_1^n \frac{k}{\sum_1^k a_i} &\le \sum_1^n \frac{4k}{k^2(k+1)^2} \sum_1^k \frac{i^2}{a_i} \\
&< 2 \sum_1^n \frac{2k+1}{k^2(k+1)^2} \sum_1^k \frac{i^2}{a_i} \\
&= 2 \sum_{k=1}^n \sum_{i=1}^k \frac{i^2}{a_i} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \\
&= 2 \sum_{i=1}^n \sum_{k=i}^n \frac{i^2}{a_i} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{i=1}^{n} \frac{i^2}{a_i}\left(\frac{1}{i^2} - \frac{1}{(n+1)^2}\right) \\
&< 2\sum_{i=1}^{n} \frac{1}{a_i}
\end{aligned}
$$