V. Cyclic Quadrilaterals

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1 All You Need To Know (sort of)

- A quadrilateral is cyclic if and only if the sum of a pair of opposite angles is 180.
- A quadrilateral is cyclic if and only if it satisfies power of a point; that is, if you let the diagonals intersect at X, then $(AX)(CX) = (BX)(DX)$. Also, if $AB \cap CD = X$, then $(AX)(BX) = (CX)(DX)$.
- A quadrilateral is cyclic if the problem says it is.
- But if the problem doesn't say a quadrilateral is cyclic, it might still be cyclic.
- And even if the problem doesn't seem to have any quadrilaterals at all, there might be a cyclic one.
- (Almost) all of these problems involve cyclic quadrilaterals.

2 Warm-Ups

1. Squat for 5 minutes straight. (note: this does not involve cyclic quadrilaterals)
2. Prove that if either of the above power-of-a-point relations hold, then the quadrilateral is cyclic.

3 Problems

1. (USAMO90) An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

   Solution: Angle chasing. $B'MC' = B'BN = 2B'BA$ since H reflects onto N over AB (previous problem). But $B'BA = C'CA$ by cyclic quads, and again that's half of $C'CQ$, and last cyclic quad sends us into $B'PC'$, which solves the problem.

2. (Razvan97) In triangle ABC, $AB = AC$. A circle tangent to the circumcircle is also tangent to AB and AC in P and Q. Prove that the midpoint M of PQ is the incenter of triangle ABC.

3. (Razvan98) Let $ABCD$ be a cyclic quadrilateral with $AC \perp BD$. Prove that the area of quadrilaterals $AOCD$ and $AOCB$ are equal, where O is the circumcenter.

   Solution: Use Brutal Force, shifting the horizontal or the vertical.

4. (Razvan97) In a circle, AB and CD are orthogonal diameters. A variable line passing through CC intersects AB in M and the circle in N. Find the locus of the intersection of the parallel to CD through M with the tangent in N.

5. (Razvan97) Let B and C be the endpoints, and A the midpoint, of a semicircle. Let M be a point on the side AC and $P, Q \in BM$, with $AP \perp BM$ and $CQ \perp BM$. Prove that $BP = PQ + QC$.

6. (Razvan97) In an inscribed quadrilateral $ABCD$, let $AB \cap BC = E$. Let $F \in AB$, $G \in CB$ such tat $CF \perp CB$, $DG \perp AD$, and let $CF \cap DG = I$. Prove that $EI \perp AB$.

7. (USAMOxx) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal, and let P be the intersection of the diagonals. Prove that the four points that are symmetric to P with respect to the sides form a cyclic quadrilateral.

8. (Razvan97) Let A, B, C be collinear and $M \notin AB$. Prove that M and the circumcenters of MAB, MBC, and MAC lie on a circle.

4 Harder Problems

1. (Razvan97) Let O be the circumcenter of triangle ABC, and AD the height. Project points B and C on AO in E and F. Let $DE \cap AC = G$, $DF \cap AB = H$, and prove that $ADGH$ is cyclic.

2. (Tucker's Circle) Prove that the six endpoints of 3 equal segments inscribed in the angles of a triangle and antiparallel with the sides lie on a circle.

   Solution: Use isogonal conjucacy; they are symmedians.

3. (MOP98) Let ω_1 and ω_2 be two circles of the same radius, intersecting at A and B. Let O be the midpoint of AB. Let CD be a chord of ω_1 passing through O, and let the segment CD meet ω_2 at P. Let EF be a chord of ω_2 passing through O, and let the segment EF meet ω_1 at Q. Prove that AB, CQ, and EP are concurrent.

   Solution: MOP98/12/3

4. (MOP98) Let D be an internal point on the side BC of a triangle ABC. The line AD meets the circumcircle of ABC again at X. Let P and Q be the feet of the perpendiculars from X to AB and AC, respectively, and let γ be the circle with diameter XD. Prove that the line PQ is tangent to γ if and only if $AB = AC$.

   Solution: MOP98/IMO2/3