V. Cyclic Quadrilaterals

Po-Shen Loh

June 24, 2003

1 All You Need To Know (sort of)

- A quadrilateral is cyclic if and only if the sum of a pair of opposite angles is 180.
- A quadrilateral is cyclic if and only if it satisfies power of a point; that is, if you let the diagonals intersect at \( X \), then \((AX)(CX) = (BX)(DX)\). Also, if \( AB \cap CD = X \), then \((AX)(BX) = (CX)(DX)\).
- A quadrilateral is cyclic if the problem says it is.
- But if the problem doesn’t say a quadrilateral is cyclic, it might still be cyclic.
- And even if the problem doesn’t seem to have any quadrilaterals at all, there might be a cyclic one.
- (Almost) all of these problems involve cyclic quadrilaterals.

2 Warm-Ups

1. Squat for 5 minutes straight. (note: this does not involve cyclic quadrilaterals)
2. Prove that if either of the above power-of-a-point relations hold, then the quadrilateral is cyclic.

3 Problems

1. (USAMO90) An acute-angled triangle \( ABC \) is given in the plane. The circle with diameter \( AB \) intersects altitude \( CC' \) and its extension at points \( M \) and \( N \), and the circle with diameter \( AC \) intersects altitude \( BB' \) and its extensions at \( P \) and \( Q \). Prove that the points \( M, N, P, Q \) lie on a common circle.
   Solution: Angle chasing. \( B'MC' = B'BN = 2B'BA \) since \( H \) reflects onto \( N \) over \( AB \) (previous problem). But \( B'BA = C'C'A \) by cyclic quads, and again that’s half of \( C'CQ \), and last cyclic quad sends us into \( B'PC' \), which solves the problem.
2. (Razvan97) In triangle \( ABC \), \( AB = AC \). A circle tangent to the circumcircle is also tangent to \( AB \) and \( AC \) in \( P \) and \( Q \). Prove that the midpoint \( M \) of \( PQ \) is the incenter of triangle \( ABC \).
3. (Razvan98) Let \( ABCD \) be a cyclic quadrilateral with \( AC \perp BD \). Prove that the area of quadrilaterals \( AOC'D \) and \( AOCD \) are equal, where \( O \) is the circumcenter.
   Solution: Use Brutal Force, shifting the horizontal or the vertical.
4. (Razvan97) In a circle, \( AB \) and \( CD \) are orthogonal diameters. A variable line passing through \( CC \) intersects \( AB \) in \( M \) and the circle in \( N \). Find the locus of the intersection of the parallel to \( CD \) through \( M \) with the tangent in \( N \).
5. (Razvan97) Let \( B \) and \( C \) be the endpoints, and \( A \) the midpoint, of a semicircle. Let \( M \) be a point on the side \( AC \) and \( P, Q \in BM \), with \( AP \perp BM \) and \( CQ \perp BM \). Prove that \( BP = PQ + QC \).
6. (Razvan97) In an inscribed quadrilateral $ABCD$, let $AB \cap BC = E$. Let $F \in AB$, $G \in CB$ such that $CF \perp CB$, $DG \perp AD$, and let $CF \cap DG = I$. Prove that $EI \perp AB$.

7. (USAMOxx) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal, and let $P$ be the intersection of the diagonals. Prove that the four points that are symmetric to $P$ with respect to the sides form a cyclic quadrilateral.

8. (Razvan97) Let $A, B, C$ be collinear and $M \notin AB$. Prove that $M$ and the circumcenters of $MAB$, $MBC$, and $MAC$ lie on a circle.

4 Harder Problems

1. (Razvan97) Let $O$ be the circumcenter of triangle $ABC$, and $AD$ the height. Project points $B$ and $C$ on $AO$ in $E$ and $F$. Let $DE \cap AC = G$, $DF \cap AB = H$, and prove that $ADGH$ is cyclic.

2. (Tucker’s Circle) Prove that the six endpoints of 3 equal segments inscribed in the angles of a triangle and antiparallel with the sides lie on a circle.

   Solution: Use isogonal conjugacy; they are symmedians.

3. (MOP98) Let $\omega_1$ and $\omega_2$ be two circles of the same radius, intersecting at $A$ and $B$. Let $O$ be the midpoint of $AB$. Let $CD$ be a chord of $\omega_1$ passing through $O$, and let the segment $CD$ meet $\omega_2$ at $P$. Let $EF$ be a chord of $\omega_2$ passing through $O$, and let the segment $EF$ meet $\omega_1$ at $Q$. Prove that $AB$, $CQ$, and $EP$ are concurrent.

   Solution: MOP98/12/3

4. (MOP98) Let $D$ be an internal point on the side $BC$ of a triangle $ABC$. The line $AD$ meets the circumcircle of $ABC$ again at $X$. Let $P$ and $Q$ be the feet of the perpendiculars from $X$ to $AB$ and $AC$, respectively, and let $\gamma$ be the circle with diameter $XD$. Prove that the line $PQ$ is tangent to $\gamma$ if and only if $AB = AC$.

   Solution: MOP98/IMO2/3