IV. Triangles

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1 Warm-up

1. (Greece) Let ABC be a triangle, O be the foot of th eangle bisector of A, and K the second intersection of AO with the circumcircle of ABC. Prove that if the incircles of BOK and COK are congruent, then ABC is isosceles.

   Solution: Brutal Force; note that K is always at the midpoint of its arc, so we are just varying the cevian from K. As the cevian deviates from perpendicular, one side squishes and one expands.

2. Let ABC be a triangle, and let ℓ be a line parallel to BC; let it intersect AB and AC at B' and C', respectively. Prove that BC' and CB' concur with the median from A.

   Solution: Affine transformation to isosceles case.

3. Let I be the incenter of triangle ABC, and A' the midpoint of the arc BC of the circumcircle. Prove that $A'B = A'C = A'I$.

4. (Kiran97) Let ABC be a triangle, and let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Let H be the orthocenter. Then:

   (a) The triangles AEF, BFD, and CDE are similar to ABC.
   Solution: Cyclic quadrilateral

   (b) H is the incenter of triangle DEF.
   Solution: Angle chasing

   (c) The reflection of H across a side of ABC lies on the circumcircle of the triangle.
   Solution: Angle chasing with cyclic quadrilateral formed by feet of altitudes.

2 Problems

1. (Nine-Point Circle) Prove that the following 9 points of triangle ABC are concyclic: the feet of the altitudes, the midpoints of AH, BH, and CH, and the midpoints of the sides. Also show that the center of this circle is the midpoint of the Euler Line.

2. (Fermat Point) Let ABC be a triangle, and construct equilateral triangles ABD, BCE, and CAF on its outside. Prove that CD, AE, and BF concur, and that they all meet with angles of 60 degrees.

   Solution: Rotation. See Geometry Revisited.

3. (Brocard Points) Show that inside any triangle ABC, there exists a unique point P such that

$$\angle PAB = \angle PBC = \angle PCA.$$

   Note: if we define the condition with the opposite orientation, we get another point; these two points are called the *Brocard Points* of ABC. The angle is called the *Brocard angle.*

Solution: The locus of points for which one of the equalities holds is a circle that is tangent to one of the sides at a vertex and passes through the appropriate other vertex. Draw in two such circles; we get one intersection. Then from these two things, we get all 3 angles equal, and so it must be the Brocard point. It is clearly uniquely determined.

4. (Brocard Angle) Let ω be the Brocard angle. Show that $\cot\omega = \cot A + \cot B + \cot C$, and that both Brocard points have the same Brocard angle.

5. (Russia) The incircle of triangle ABC touches sides AB, BC, and CA at M, N, K, respectively. The line through A parallel to NK meets MN at D. The line through A parallel to MN meets NK at E. Show that the line DE bisects sides AB and AC of triangle ABC.

   Solution: Let the lines AD and AE meet BC at F and H, respectively. It suffices to show that D and E are the midpoints of AF and AH, respectively. Since $BN = BM$ and $MN \parallel AH$, the trapezoid $AMNH$ is isosceles, so $NH = AM$. Likewise $NF = AK$. Since $AK = AM$, N is the midpoint of FH. Since NE is parallel to AF, E is the midpoint of AH, and likewise D is the midpoint of AF.

6. (Greece) Let ABC be an acute triangle, AD, BE, CZ its altitudes and H its orthocenter. Let AI, $A\Theta$ be the internal and external bisectors of angle A. Let M, N be the midpoints of BC, AH, respectively. Prove that

   (a) MN is perpendicular to EZ;

   (b) if MN cuts the segments AI, $A\Theta$ at the points K, L, then $KL = AH$.

   Solution: The circle with diameter AH passes through Z and E, and so $ZN = NE$. On the other hand, MN is a diameter of the nine-point circle of ABC, and Z and E lie on that circle, so $ZN = ZE$ implies that $ZE \perp MN$.

   As determined in (a), MN is the perpendicular bisector of segment ZE. The angle bisector AI of $\angle EAZ$ passes through the midpoint of the minor arc EZ, which clearly lies on MN; therefore this midpoint is K. By similar reasoning, L is the midpoint of the major arc EZ. Thus KL is also a diameter of circle EAZ, so KL = MN.

7. (Hungary) Let R be the circumradius of triangle ABC, and let G and H be its centroid and orthocenter, respectively. Let F be the midpoint of GH. Show that $AF^2 + BF^2 + CF^2 = 3R^2$.

   Solution: Use vectors with the origin at the circumcenter. Then $G = (A + B + C)/3$ and $H = A + B + C$. So $F = (2/3)(A + B + C)$. Now just hack away with $(A - F)\cdot(A - F) + (B - F)\cdot(B - F) + (C - F)\cdot(C - F)$.

8. (MOP98) The altitudes through vertices A, B, C of acute triangle ABC meet the opposite sides at D, E, F respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of triangle PQR passes through the midpoint of BC.

   Solution: MOP98/5/1; uses nine point circle

9. (MOP98) Let G be the centroid of triangle ABC, and let R, R_1, R_2, R_3 denote the circumradii of triangles ABC, GAB, GBC, GCA, respectively. Prove that $R_1 + R_2 + R_3 \geq 3R$.

   Solution: MOP98/8/3

10. (Kiran97) Let ℓ be a line through the orthocenter H of a triangle ABC. Prove that the reflections of ℓ across AB, BC, and CA all pass through a common point; show also that this point lies on the circumcircle of ABC.

    Solution: First show that any two reflections concur with the circle. (angle chasing). It's easier to start by showing that the reflections corresponding to the sides of the triangle that ℓ intersects work. Make sure to use the fact that orthocenter is reflected onto the circumcircle.

3 Harder Problems

1. (Morley) The intersections of adjacent trisectors of the angles of a triangle are the vertices of an equilateral triangle.

2. (MOP03) Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

   if and only if P lies on at least one of the medians of triangle ABC.

   Solution: Start by observing that our condition is equivalent to:

$$[DAB] + [EBC] + [FCA] = \frac{3}{2}[ABC]$$

3. (USAMO90) An acute triangle ABC is given. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

   Solution: Angle chasing. $B'MC' = B'BN = 2B'BA$ since H reflects onto N over AB (previous problem). But $B'BA = C'CA$ by cyclic quads, and again that's half of $C'CQ$, and last cyclic quad sends us into $B'PC'$, which solves the problem.

4. (Razvan98) Let ABC be a triangle and D, E, F the points of tangency of the sides BC, AC, and AB with the incircle. Let $T \in EF$ and $Q \in DF$ such that $EQ \parallel DT \parallel AB$. Prove that CF, DE, and QT intersect.

   Solution: Using a problem from above, it suffices to show that CF is the median of a triangle with vertex F and base parallel to AB. Construct such a beast with base through C. Now, let the base angle on the same side as A be A' and the other one be B'. It suffices to show that $B'C = DC$ and $DC = A'C$, because common tangents tell us that $DC = EC$. But angle case; by parallel lines, the base angle at A' is equal to angle AFE, and by isoceles, that's AEF. Vertical angles tell us that it is $A'EC$, which solves our problem.

5. (China) Let H be the orthocenter of acute triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, H are collinear.

   Solution: The line PQ is the polar of A with respect to the circle, so it su ces to show that A lies on the pole of H. Let D and E be the feet of the altitudes from A and B, respectively; these also lie on the circle, and $H = AD \cap BE$. The polar of the line AD is the intersection of the tangents AA and DD, and the polar of the line BE is the intersection of the tangents BB and EE. The collinearity of these two intersections with C = $AE \cap BD$ follows from applying Pascal's theorem to the cyclic hexagons AABDDE and ABBDEE. (An elementary solution with vectors is also possible and not difficult.)

4 Really Harder Problems

1. End world hunger.

2. Take a shower without someone flushing the toilet.

3. Stop Yan's contagious disease.