1 Problems

Putnam 2003/A4. Suppose that $a, b, c, A, B, C$ are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers $x$. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$ 

Putnam 2003/A5. A Dyck $n$-path is a lattice path of $n$ upsteps $(1, 1)$ and $n$ downsteps $(1, -1)$ that starts at the origin $O$ and never dips below the $x$-axis. A return is a maximal sequence of contiguous downsteps that terminates on the $x$-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.

Show that there is a one-to-one correspondence between the Dyck $n$-paths with no return of even length and the Dyck $(n - 1)$-paths.

Putnam 2003/A6. For a set $S$ of non-negative integers, let $r_S(n)$ denote the number of ordered pairs $(s_1, s_2)$ such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the non-negative integers into two sets $A$ and $B$ in such a way that $r_A(n) = r_B(n)$ for all $n$?