2. Polynomials
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1 Classical results

Algebra. If $r$ is a root of the polynomial $P(x)$, then $P$ factors as $(x - r)Q(x)$ for some polynomial $Q$.

Algebra. Every polynomial of degree $n$ has at most $n$ distinct roots.

Lagrange Interpolation. Show that there is a degree-4 polynomial which takes values $P(0) = 0$, $P(1) = 0$, $P(2) = 0$, $P(3) = 1$, and $P(4) = 1$.

Reed-Solomon codes. Automatic spell checkers know to correct “teh” to “the”. More abstractly, an error-correcting code with minimum distance $d$ is a collection of strings of length $n$ from an alphabet $A$, with the property that any two strings differ by at least $d$ pointwise edits. It turns out that there are nice error-correcting codes with minimum distance $d$ over alphabets of size $q$, for prime powers $q$, and these are based on polynomials!

Multiple roots. If $r$ is a real root of the polynomial $P(x)$, and $r$ has multiplicity greater than 1, then both $P(r) = 0$ and $P'(r) = 0$.

Gauss-Lucas. The zeros of the derivative $P'(z)$ of any polynomial lie in the convex hull of the zeros of the polynomial $P(z)$.

2 Problems

1. Find a polynomial with integer coefficients that has the zero $\sqrt{2} + \sqrt{3}$.

2. There is no polynomial which has the property that $P(k) = 2^k$ for all positive integers $k$.

3. Let $a_1, \ldots, a_n$ be positive real numbers. Prove that the polynomial $P(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n$ has a unique positive zero.

4. Solve the system

\[
\begin{align*}
  x + y + z &= 1 \\
  xyz &= 1
\end{align*}
\]

knowing that $x, y, z$ are complex numbers of absolute value equal to 1.

5. Let $P(z)$ and $Q(z)$ be polynomials with complex coefficients of degree greater than or equal to 1 with the property that $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Prove that the polynomials are equal.
6. Let \( P(x) \) and \( Q(x) \) be arbitrary polynomials with real coefficients, and let \( d \) be the degree of \( P(x) \). Assume that \( P(x) \) is not the zero polynomial. Prove that there exist polynomials \( A(x) \) and \( B(x) \) with real coefficients, such that:

(i) both \( A \) and \( B \) have degree at most \( d/2 \), and

(ii) at most one of \( A \) and \( B \) is the zero polynomial, and

(iii) \( \frac{A(x)+Q(x)B(x)}{P(x)} \) is a polynomial with real coefficients. That is, there is some polynomial \( C(x) \) with real coefficients such that \( A(x) + Q(x)B(x) = P(x)C(x) \).

3 Homework

Please write up solutions to two of the problems, to turn in at next week’s meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.