

21-228 Discrete Mathematics

Assignment 7

Due Fri Apr 19, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Is the following statement true for every positive integer n ?

Every graph consisting of two edge-disjoint Hamiltonian paths contains a Hamiltonian cycle.

2. Construct a random graph on n vertices as follows. Start with n vertices, with no edges. Then, for each of the $\binom{n}{2}$ pairs of vertices, flip a (very unfair) coin which is heads with probability $\frac{1}{n^{1.01}}$. If it is heads, put the edge between that pair of vertices; otherwise, do not.

Let q_n be the probability that the resulting graph contains a triangle (three vertices a, b, c with edges ab , bc , and ca all present). Show that

$$\lim_{n \rightarrow \infty} q_n = 0.$$

3. Construct a random graph on n vertices as follows. Start with n vertices, with no edges. Then, for each of the $\binom{n}{2}$ pairs of vertices, flip a (very unfair) coin which is heads with probability $\frac{1}{n}$. If it is heads, put the edge between that pair of vertices; otherwise, do not. Prove that the probability of the graph being connected tends to zero as $n \rightarrow \infty$. (Much more is true.)
4. Let $R(t, t, t)$ be the smallest integer n such that every 3-coloring of the edges of K_n contains a monochromatic K_t . Prove that

(a) $R(t, t, t) \leq 27^t$, and

- (b) derive an exponential lower bound for the 3-color Ramsey number $R(t, t, t)$. That is, for each t , come up with an n_t , and a 3-coloring of the edges of K_{n_t} such that there is no monochromatic K_t , where n_t grows exponentially with t .

Note: I can do this with $n_t = 3^{t/2}$ for every even t (which is sufficient), but it is OK if you prove this with a different n_t , as long as yours grows exponentially with t .

5. There are 100 cities in a country, called “1”, “2”, \dots , “100”. The government would like to connect them all with a network, and it costs $\max\{|i - j|, 4\}$ to lay a wire between cities “ i ” and “ j ”. (There is a minimum cost of 4 per wire.) Another company comes along, offering to create microwave links between city pairs at a cost of 2 each, with the catch that their technology can only connect city pairs of the form (“ i ”, “ $2i$ ”). For example, it would cost 30 to lay a wire between cities 31 and 61, 4 to lay a wire between cities 31 and 32, and 2 to microwave-link 30 and 60. What is the minimum cost to create this connected network?