1 Problems

**Putnam 2003/B1.** Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that
\[1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)\]
holds identically?

**Putnam 2003/B2.** Let $n$ be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n - 1$ entries \(\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}\) by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n - 2$ entries, and continue until the final sequence produced consists of a single number $x_n$. Show that $x_n < 2/n$.

**Putnam 2003/B3.** Show that for each positive integer $n$,
\[n! = \prod_{i=1}^{n} \text{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.\]
(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)