7. Convergence
Po-Shen Loh
CMU Putnam Seminar, Fall 2018

1 Well-known statements

Infimum and supremum. The infimum of a set of numbers $S \subseteq \mathbb{R}$ is the largest real number $x$ such that $x \leq s$ for all $s \in S$. The supremum is the smallest real number $y$ such that $y \geq s$ for all $s \in S$.

Monotone sequence. Let $a_1 \geq a_2 \geq a_3 \geq \cdots$ be a sequence of non-negative numbers. Then $\lim_{n \to \infty} a_n$ exists.

Alternating series. Let $a_1, a_2, \ldots$ be a sequence of real numbers which is decreasing in absolute value, converging to 0, and alternating in sign. Then the series $\sum_{i=1}^{\infty} a_i$ converges.

Conditionally convergent. Let $a_1, a_2, \ldots$ be a sequence of real numbers such that the series $\sum_{i=1}^{\infty} a_i$ converges to a finite number, but $\sum_{i=1}^{\infty} |a_i| = \infty$. This is called a conditionally convergent series. Then, for any real number $r$, the sequence can be rearranged so that the series converges to $r$. (Formally, there exists a bijection $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $\lim_{n \to \infty} \sum_{i=1}^{n} a_{\sigma(i)} = r$.)

2 Problems

1. Which of these these series converge or diverge: $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{2^n}$, $\sum_{n=1}^{\infty} \frac{1}{n \log n}$, $\sum_{n=1}^{\infty} \frac{1}{n \log n \log \log n}$, …?

2. Prove that the equation $x^{x^{\ldots}} = 2$ is satisfied by $x = \sqrt{2}$, but that the equation $x^{x^{\ldots}} = 4$ has no solution. What is the "break-point"?

3. The expression $\sqrt{-\frac{2}{9}} + \sqrt{-\frac{2}{9} + \sqrt{-\frac{2}{9} + \cdots}}$ is not clearly defined. Consider the recursion $a_{n+1} = \sqrt{-\frac{2}{9} + a_n}$ with $a_1 = 0.34$. What does that sequence converge to, and why?

4. Suppose that $a_1, a_2, \ldots$ is a sequence of real numbers such that for each $k$, $a_k$ is either $\frac{1}{2}$ or $-\frac{1}{2}$, and $a_k$ has the same sign as $a_{k+8}$. Show that if four of $a_1, a_2, \ldots, a_8$ are positive, then $\sum_{k=1}^{\infty} a_k$ converges. Is the converse true?

5. Prove that a sequence of positive numbers, each of which is less than the average of the previous two, is convergent.

6. Prove that every sequence of real numbers contains a monotone subsequence. Formally, show that for every sequence $a_1, a_2, \ldots$ of real numbers, there is a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(i) < f(j)$ for all $i < j$, and either $a_{f(1)} \leq a_{f(2)} \leq a_{f(3)} \leq \cdots$ or $a_{f(1)} \geq a_{f(2)} \geq a_{f(3)} \geq \cdots$.

7. Given a convergent series of positive terms, $\sum_{k=1}^{\infty} a_k$, does the series $\sum_{k=1}^{\infty} \frac{a_1 + \cdots + a_k}{k}$ always converge? How about the series $\sum_{k=1}^{\infty} \sqrt[n]{a_1 \times \cdots \times a_k}$?
8. Let \(a_1, a_2, \ldots\) be a sequence of positive real numbers satisfying \(a_n < a_{n+1} + a_{n^2}\) for all \(n\). Prove that \(\sum a_n\) diverges.

9. Let \(a_i\) be real numbers such that \(\sum_{i=1}^{\infty} a_i = 1\) and \(|a_1| > |a_2| > |a_3| > \cdots\). Suppose that \(f\) is a bijection from the positive integers to itself, and

\[|f(i) - i||a_i| \to 0\]

as \(i \to \infty\). Prove or disprove that \(\sum_{i=1}^{\infty} a_{f(i)} = 1\).

3. **Homework**

Please write up solutions to two of the problems, to turn in at next week’s meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.