3. Number theory
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1 Well-known statements

**Fermat’s Little Theorem.** For every prime $p$ and any integer $a$ which is not divisible by $p$, we have $a^{p-1} \equiv 1 \pmod{p}$.

**Euler’s Theorem.** Let $\varphi(n)$ denote the number of positive integers in $\{1, 2, \ldots, n\}$ which are relatively prime to $n$. Then, for any integer $a$ which is relatively prime to $n$,

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$ 

**Wilson’s Theorem.** A positive integer $n$ is a prime if and only if $(n-1)! \equiv -1 \pmod{n}$.

**Dirichlet’s Theorem.** For any two positive integers $a$ and $d$ which are relatively prime, the arithmetic progression $a, a+d, a+2d, \ldots$ contains infinitely many primes.

**Quadratic residues.** Let $p$ be a prime. There are exactly $\frac{p+1}{2}$ residues $r$ such that there exist solutions to $x^2 \equiv r \pmod{p}$.

2 Problems

1. The 9-digit number $2^{29}$ has exactly 9 digits, and they are all distinct. Which of the 10 possible digits 0–9 does not appear?

2. There are infinitely many primes of the form $4n - 1$, where $n$ is an integer.

3. Let $p$ be a prime, and let $n \geq k$ be non-negative integers. Prove that

$$\binom{pn}{pk} \equiv \binom{n}{k} \pmod{p}.$$ 

4. Show that for every positive integer $n$, there is an integer $N > n$ such that the number $5^n$ appears as the last few digits of $5^N$. For example, if $n = 3$, we have $5^3 = 125$, and $5^5 = 3125$, so $N = 5$ would work.

5. Prove that the product of 3 consecutive integers is never a perfect power (i.e., a perfect square, a perfect cube, etc).

6. How many integers $r$ in $\{0, 1, \ldots, 2^n - 1\}$ are there for which there exists an $x$ where $x^2 \equiv r \pmod{2^n}$?

7. Let $n, a, b$ be positive integers. Prove that $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$.

8. A positive integer is written at each integer point in the plane $(\mathbb{Z}^2)$, in such a way that each of these numbers is the arithmetic mean of its four neighbors. Prove that all of the numbers are equal.

9. A triangular number is a positive integer of the form $n(n+1)/2$. Prove that $m$ is the sum of two triangular numbers if and only if $4m + 1$ is the sum of two squares.
3 Homework

Please write up solutions to two of the problems, to turn in at next week’s meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.