2. Polynomials

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1 Well-known statements

Limited roots. A polynomial of degree n has exactly n (complex) roots, counted with multiplicity.

Complete factorization. If the n roots of a degree-n polynomial $p(z)$ are $r_1, \ldots, r_n$, then we can express $p(z)$ as $a(z - r_1) \cdots (z - r_n)$.

Vieta's formulas. If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, then the product of the roots is $(-1)^n a_0$, and the sum of the roots is $-a_{n-1}$.

Uniqueness. For each nonnegative integer n, if two polynomials $p(x) = a_n x^n + \cdots + a_0$ and $q(x) = b_n x^n + \cdots + b_0$ are equal for $n+1$ distinct values of x, then all of their coefficients are equal, and they are the same polynomial.

Lagrange interpolation. If $p(x)$ is a polynomial of degree n, and we have real numbers $x_1, \ldots, x_{n+1}$ and $y_1, \ldots, y_{n+1}$ such that every $p(x_i) = y_i$, then there is an explicit formula for the polynomial $p(x)$.

2 Problems

1. If 3 distinct points on the curve $y = x^3$ are collinear, then the sum of the x-coordinates of those 3 points equals 0. There's actually a similar Putnam problem: show that if 4 distinct points on the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear, then their average x-coordinate is some constant k; determine k.

2. An *elliptic curve* is the set of points (x, y) satisfying the equation $y^2 = x^3 + ax + b$. A point (x, y) on the curve is *rational* if and only if both x and y are rational. Prove that given any two rational points on the curve, the line through them intersects the curve at most one more time, and if it intersects one more time, that intersection point is also rational.

3. Let $P(z)$ be a polynomial with complex coefficients. Prove that $P(z)$ is an even function if and only if there exists a polynomial $Q(z)$ with complex coefficients satisfying $P(z) = Q(z)Q(-z)$.

4. Describe all ordered pairs (a, b) of real numbers such that both (possibly complex) roots of $z^2 + az + b = 0$ satisfy $|z| < 1$.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that for each fixed x, the function $g_x(y) = f(x, y)$ is a polynomial in y, and for each fixed y, the function $h_y(x) = f(x, y)$ is a polynomial in x. Prove that $f(x, y)$ must be a polynomial in x and y, i.e., that there is a finite n, and a finite collection of real numbers $a_{j,k}$ with $0 \le j, k \le n$, such that $f(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k} x^j y^k$.

6. Prove that there is a function $f : \mathbb{Q}^2 \to \mathbb{Q}$ with the above property, but which is not a polynomial in x and y.

7. Invent a single (binary) operation $\star$ such that for every real numbers a and b, the operations $a + b$, $a - b$, $a \times b$, and $a \div b$ can be created by applying just $\star$ (possibly many times), starting with just a's and b's.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.