5. Functional Equations

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1 Classical results

Cauchy. Linear functions through the origin are the only continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

2 Problems

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = \frac{1}{2}$, and there is some real α for which

$$f(x + y) = f(x)f(\alpha - y) + f(y)f(\alpha - x)$$

for all x, y. Prove that f is constant.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x + y) + f(y + z) + f(z + x) \geq 3f(x + 2y + 3z)$$

for all x, y, z.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Prove that f is constant.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function. Prove that the system

$$x = f(y)\,, \quad y = f(z)\,, \quad z = f(x)$$

has a unique solution.

5. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$[f(x) + f(y)]\,[f(u) + f(v)] = f(xu - yv) + f(xv + yu)$$

for all x, y, u, v.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(0) = 0$, $f(1) = 1$, and $f(f(f(f(x)))) = x$ for every $x \in [0, 1]$. Prove that $f(x) = x$ for each $x \in [0, 1]$.

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + f(x)^2$$

for all x, y.

8. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x) + f(y) + f(x)f(y)$$

for all real $x, y \in \mathbb{R}$.

9. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for all n.

10. Does there exist a function f from the positive integers to the positive integers such that $f(1) = 2$, $f(f(n)) = f(n) + n$ for all n, and $f(n) < f(n+1)$ for all n?

11. (From current research.) Define the recursion:

$$\begin{aligned} \ell_0(s) &= e^{-s} \\ \ell_{t+1}(s) &= \frac{1}{1+\ell_t(\frac{1}{2})} \left[\begin{array}{l} \ell_t\left(s \cdot \frac{1+\ell_t(\frac{1}{2})}{2}\right)^2 - \ell_t\left(s \cdot \frac{1+\ell_t(\frac{1}{2})}{2}\right) \ell_t\left(\frac{1}{2} + s \cdot \frac{1+\ell_t(\frac{1}{2})}{2}\right) \\ +\ell_t\left(\frac{1}{2} + s \cdot \frac{1+\ell_t(\frac{1}{2})}{2}\right) + \ell_t\left(\frac{1}{2}\right) \ell_t\left(s \cdot \frac{1+\ell_t(\frac{1}{2})}{2}\right) \end{array} \right] \end{aligned}$$

Prove that $\ell_t(\frac{1}{2}) \to 1$ as $t \to \infty$.[1]

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.

[1] P. Loh and E. Lubetzky, Stochastic coalescence in logarithmic time, *Annals of Applied Probability*, to appear.