Putnam $\Sigma.6$

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1 Problems

Putnam 1983/B4. Let $f(n) = n + \lfloor \sqrt{n} \rfloor$. Prove that for any positive integer m, the sequence m, $f(m)$, $f(f(m))$, $f(f(f(m)))$, $\ldots$, contains at least one perfect square.

Putnam 1983/B5. Let $\|x\|$ denote the distance from x to the nearest integer. Determine

$$\frac{1}{n} \int_1^n \|n/x\| dx.$$

You may assume that

$$\prod_1^\infty \frac{2n}{2n-1} \frac{2n}{2n+1} = \frac{\pi}{2}.$$

Putnam 1983/B6. Let α be a complex (2^n+1)-th root of unity. Prove that there always exist polynomials $p(x)$ and $q(x)$ with integer coefficients, such that

$$p(\alpha)^2 + q(\alpha)^2 = -1.$$