# Putnam Σ.4

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## 1 Problems

**Putnam 1984/B4.** Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ such that (i) for every $x > 0$, $f(x) > 0$, and (ii) for all $a > 0$, the centroid of the region under the curve $y = f(x)$ between $0 \leq x \leq a$ has $y$-coordinate equal to the average value of $f(x)$ on $[0, a]$.

**Putnam 1984/B5.** Let $f(n)$ be the number of 1's in the binary expression for $n$. Let $g(m) = \pm 0^m \pm 1^m \pm 2^m \pm \cdots \pm (2^m - 1)^m$, where we take the sign + for $k^m$ iff $f(k)$ is even. Show that $g(m)$ can be written in the form $(-1)^m a^{p(m)}(q(m))!$ where $a$ is an integer and $p$ and $q$ are polynomials.

**Putnam 1984/B6.** Define a sequence of convex polygons $P_n$ as follows. $P_0$ is an equilateral triangle of side 1, and $P_{n+1}$ is obtained from $P_n$ by cutting off the corners 1/3 of the way along each side. For example, $P_1$ is a regular hexagon of side 1/3. Find $\lim_{n\to\infty} \text{area}(P_n)$.