14. Bare hands

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1 Well-known results

Fast matrix multiplication. It's possible to multiply two $n \times n$ matrices using only $O(n^c)$ multiplications, where $c < 3$.

2 Problems

1. Derive the operations $+$, $-$, $\times$, and $\div$ from $-$ and reciprocal.
2. Invent a single (binary) operation from which $+$, $-$, $\times$, and $\div$ can be derived.
3. The multiplication of two complex numbers $(a + bi) \cdot (x + yi) = (ax - by) + (bx + ay)i$ appears to need 4 real multiplications (ax, by, bx, ay), but does it really? If additions are free, can this same job be accomplished in 3 real multiplications? In 2?
4. At a certain corner, the traffic light is green for 30 seconds and then red for 30 seconds. On average, how much time is lost at this corner?
5. Prove that there is no equilateral triangle all of whose vertices are plane lattice points. (How about 3D lattice points?)
6. Let α be a complex $(2^n + 1)$-th root of unity. Prove that there always exist polynomials $p(x)$ and $q(x)$ with integer coefficients, such that
$$p(\alpha)^2 + q(\alpha)^2 = -1.$$
7. Compute the inverse of the following matrix:
$$\begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n}{0} \\ \binom{0}{1} & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{n}{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{0}{n} & \binom{1}{n} & \binom{2}{n} & \cdots & \binom{n}{n} \end{pmatrix},$$
where $\binom{n}{k} = 0$ when $n < k$.

3 Homework

Please write up solutions to two of the statements/problems, to turn in at next week's meeting. One of them may be a problem that we solved in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.