1. Introduction

Po-Shen Loh

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1 Well-known statements

Well-ordering principle. Every non-empty set of positive integers contains a minimum element.

Place value. There are 6 buckets in a row, with one penny in each bucket. You have one magic move, which you can perform as many times as you wish: you may remove one penny from a bucket, and add two pennies to the bucket to its immediate right. (If you removed a penny from the rightmost bucket, no pennies are added.) You may stop at any time. How many pennies can you end up with in total over the 6 buckets?

Fermat's Last Theorem for $n = 4$. The equation $a^4 + b^4 = c^4$ has no positive integer solutions.

2 Problems

1. There are 6 buckets in a row, with one penny in each bucket. You have two magic moves, which you may perform as many times as you wish. Move A allows you to remove one penny from a bucket, and add two pennies to the bucket to its immediate right. (If you removed a penny from the rightmost bucket, no pennies are added.) Move B allows you to remove one penny from a bucket, and swap the contents of the two buckets to that bucket's immediate right. (If you removed a penny from the rightmost bucket or the second-rightmost bucket, then no swapping happens.) You may stop at any time. How many pennies can you end up with in total over the 6 buckets?

2. Prove that the number of pennies in the problem above cannot go to infinity.

3. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite increasing sequence of positive integers. Prove that there is one and exactly one integer $n \geq 1$ such that
$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leq z_{n+1}.$$

4. Show that $(36x + y)(x + 36y)$ is not a power of 2 for any positive integers x and y.

5. Calculate $\prod_{k=2}^{\infty} \frac{k^3-1}{k^3+1}$.

6. Let $n > 1$ be an odd integer, and let $\omega = e^{\pi i/n}$. Find integers $a_0, a_1, \ldots, a_n$ such that $\sum a_k \omega^k = \frac{1}{1-\omega}$.

7. Prove that $\cos^{-1} \frac{1}{3}$ is an irrational multiple of π. (We are working in radians.)

8. Let S be the set of all triples (x, y, z) of positive irrational numbers (not necessarily distinct) such that $x + y + z = 1$. Given a triple $P = (x, y, z) \in S$, let $P' = (\{2x\}, \{2y\}, \{2z\})$, where $\{t\}$ denotes the fractional part of t, i.e., $\{1.258\} = 0.258$. Does repeated application of this operation necessarily produce a triple with all elements $< \frac{1}{2}$, no matter what triple from S one starts with?

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.