1 Problems

Putnam 2009/A1. Let \( f \) be a real-valued function on the plane such that for every square \( ABCD \) in the plane, \( f(A) + f(B) + f(C) + f(D) = 0 \). Does it follow that \( f(P) = 0 \) for all points \( P \) in the plane?

Putnam 2009/A2. Functions \( f, g, h \) are differentiable on some open interval around 0 and satisfy the equations and initial conditions

\[
\begin{align*}
    f' &= 2f^2 gh + \frac{1}{gh}, \quad f(0) = 1, \\
    g' &= fg^2 h + \frac{4}{fgh}, \quad g(0) = 1, \\
    h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.
\end{align*}
\]

Find an explicit formula for \( f(x) \), valid in some open interval around 0.

Putnam 2009/A3. Let \( d_n \) be the determinant of the \( n \times n \) matrix whose entries, from left to right and then from top to bottom, are \( \cos 1, \cos 2, \ldots, \cos(n^2) \). For example, \( d_3 \) is the determinant of the matrix

\[
\begin{pmatrix}
    \cos 1 & \cos 2 & \cos 3 \\
    \cos 4 & \cos 5 & \cos 6 \\
    \cos 7 & \cos 8 & \cos 9
\end{pmatrix}.
\]

Note: The argument of cosine is always in radians, not degrees. Evaluate \( \lim_{n \to \infty} d_n \).

Putnam 2009/A4. Let \( S \) be a set of rational numbers such that

(a) \( 0 \in S \);
(b) If \( x \in S \), then \( x + 1 \in S \) and \( x - 1 \in S \); and
(c) If \( x \in S \) and \( x \neq \{0, 1\} \), then \( \frac{1}{x(x-1)} \in S \).

Must \( S \) contain all rational numbers?

Putnam 2009/A5. Is there a finite abelian group \( G \) such that the product of the orders of all its elements is \( 2^{2009} \)?

Putnam 2009/A6. Let \( f : [0,1]^2 \to \mathbb{R} \) be a continuous function on the closed unit square such that \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist and are continuous on the interior \( (0,1)^2 \). Let \( a = \int_0^1 f(0,y)dy, b = \int_0^1 f(1,y)dy, c = \int_0^1 f(x,0)dx \), and \( d = \int_0^1 f(x,1)dx \). Prove or disprove: There must be a point \( (x_0, y_0) \) in \( (0,1)^2 \) such that

\[
\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.
\]