1 Problems

Putnam 1986/A4. A transversal of an \( n \times n \) matrix \( A \) consists of \( n \) entries of \( A \), no two in the same row or column. Let \( f(n) \) be the number of \( n \times n \) matrices \( A \) satisfying the following two conditions:

(a) Each entry \( a_{i,j} \) of \( A \) is in the set \( \{-1, 0, 1\} \).
(b) The sum of the \( n \) entries of a transversal is the same for all transversals of \( A \).

An example of such a matrix \( A \) is

\[
A = \begin{pmatrix}
-1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

Determine with proof a formula for \( f(n) \) of the form

\[
f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,
\]

where the \( a_i \)'s and \( b_i \)'s are rational numbers.

Putnam 1986/A5. Suppose \( f_1(x), f_2(x), \ldots, f_n(x) \) are functions of \( n \) real variables \( x = (x_1, \ldots, x_n) \) with continuous second-order partial derivatives everywhere on \( \mathbb{R}^n \). Suppose further that there are constants \( c_{ij} \) such that

\[
\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}
\]

for all \( i \) and \( j \), \( 1 \leq i \leq n \), \( 1 \leq j \leq n \). Prove that there is a function \( g(x) \) on \( \mathbb{R}^n \) such that \( f_i + \partial g/\partial x_i \) is linear for all \( i \), \( 1 \leq i \leq n \). Recall that a linear function is one of the form

\[
a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.
\]

Putnam 1986/A6. Let \( a_1, a_2, \ldots, a_n \) be real numbers, and let \( b_1, b_2, \ldots, b_n \) be distinct positive integers. Suppose that there is a polynomial \( f(x) \) satisfying the identity

\[
(1 - x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.
\]

Find a simple expression (not involving any sums) for \( f(1) \) in terms of \( b_1, b_2, \ldots, b_n \) and \( n \) (but independent of \( a_1, a_2, \ldots, a_n \)).