1 Problems

Putnam 1989/B4. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

Putnam 1989/B5. Label the vertices of a trapezoid $T$ (quadrilateral with two parallel sides) inscribed in the unit circle as $A, B, C, D$ so that $AB$ is parallel to $CD$ and $A, B, C, D$ are in counterclockwise order. Let $s_1, s_2,$ and $d$ denote the lengths of the line segments $AB, CD,$ and $OE,$ where $E$ is the point of intersection of the diagonals of $T,$ and $O$ is the center of the circle. Determine the least upper bound of $\frac{s_1 - s_2}{d}$ over all such $T$ for which $d \neq 0,$ and describe all cases, if any, in which it is attained.

Putnam 1989/B6. Let $(x_1, x_2, \ldots, x_n)$ be a point chosen at random from the $n$-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1.$ Let $f$ be a continuous function on $[0, 1]$ with $f(1) = 0.$ Set $x_0 = 0$ and $x_{n+1} = 1.$ Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n}(x_{i+1} - x_i)f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt,$ where $P$ is a polynomial of degree $n,$ independent of $f,$ with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1.$