7. Convergence

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1 Famous results

Continued root. \( \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}} = 3. \)

Supremum/infinum. The \textit{supremum} of a set \( S \) of real numbers is defined to be the smallest real number \( y \) such that all \( s \in S \) are less than or equal to \( y \). The \textit{infimum} is the largest real number \( x \) such that all \( s \in S \) are greater than or equal to \( x \). A bounded sequence of monotonically increasing real numbers always converges to its supremum.

Limit superior/inferior.

\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \left( \inf_{k \geq n} x_k \right) ; \quad \limsup_{n \to \infty} x_n = \lim_{n \to \infty} \left( \sup_{k \geq n} x_k \right) .
\]

The \( \liminf \) of a sequence is always less than or equal to its \( \limsup \), and if they are equal, then the sequence has a limit.

Sub-additivity. Let \( x_1, x_2, \ldots \) be a sequence of real numbers such that \( x_{i+j} \leq x_i + x_j \) for all (not necessarily distinct) positive integers \( i \) and \( j \). Then \( \lim_{n \to \infty} \frac{1}{n} x_n \) always exists, and is either a real number or \(-\infty\).

2 Problems

1. Let \( a_1, a_2, \ldots \) be a sequence of non-negative real numbers such that \( a_{m+n} \leq a_m a_n \) for all \( m, n \in \mathbb{Z}^+ \). (This even includes cases when \( m = n \).) Show that the sequence \( a_n^{1/n} \) converges.

2. For each \( n \), let \( f(n) \) denote the largest integer such that \( 2^{f(n)} \) divides \( n \). For example, \( f(3) = 0 \) since 3 is odd, and \( f(24) = 3 \) since \( 2^3 \) is the highest power of 2 which divides 24. Let \( g(n) = f(1) + f(2) + \cdots + f(n) \). Prove that \[
\sum_{n=1}^{\infty} e^{-g(n)}
\]

converges.

3. Let \( a_1, a_2, \ldots \) be a sequence of real numbers such that the sequence \( a_1 + 2a_2, a_2 + 2a_3, a_3 + 2a_4, \ldots \) converges. Prove that the sequence \( a_1, a_2, \ldots \) must then also converge.

4. What if we are told that the sequence \( a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots \) converges? Does that imply that \( a_1, a_2, \ldots \) converges? For a third variant, must \( a_1, a_2, \ldots \) converge if we are told that \( 2a_1 + a_2, 2a_2 + a_3, \ldots \) converges?

5. Let \( a_1, a_2, \ldots \) be a sequence of non-negative real numbers, for which \( \sum a_i \) converges. Suppose also that \( a_j \leq 100a_i \) for all \( i \leq j \leq 2i \). Show that \( \lim_{n \to \infty} na_n = 0 \).
6. Let $a_1, a_2, \ldots$ be a strictly increasing sequence of positive real numbers, such that $\sum a_i^{-1}$ converges. For each positive real $x$, let $f(x)$ be the largest integer $i$ for which $a_i < x$. Prove that

$$\lim_{x \to \infty} \frac{f(x)}{x} = 0.$$ 

7. Let

$$a_n = \sqrt{1 + 2} \sqrt{1 + 3} \sqrt{1 + 4} \cdots \sqrt{1 + (n - 3)} \sqrt{1 + (n - 2)} \sqrt{1 + (n - 1)} \sqrt{1 + n}.$$ 

Prove that $\lim_{n \to \infty} a_n = 3.$

3 Homework

Please write up solutions to two of the problems, to turn in at next week’s meeting. One of them may be a problem that we discussed in class.