21-228 Discrete Mathematics

Assignment 7

Due Mon Apr 22, at start of class

Notes: Collaboration is permitted until the writing stage. Please justify every numerical answer with an explanation. The base value of each question is 10 points.

1. Construct a random graph on n vertices as follows. Start with n vertices, with no edges. Then, for each of the $\binom{n}{2}$ pairs of vertices, flip a (very unfair) coin which is heads with probability $\frac{1}{n^{1.01}}$. If it is heads, put the edge between that pair of vertices; otherwise, do not.

   Let q_n be the probability that the resulting graph contains a triangle (three vertices a, b, c with edges ab, bc, and ca all present). Show that

   $$\lim_{n\to\infty} q_n = 0\,.$$

2. Construct a random graph on n vertices as follows. Start with n vertices, with no edges. Then, for each of the $\binom{n}{2}$ pairs of vertices, flip a (very unfair) coin which is heads with probability $\frac{1}{n}$. If it is heads, put the edge between that pair of vertices; otherwise, do not. Prove that the probability of the graph being connected tends to zero as $n \to \infty$. (Much more is true.)

3. Let $R(t, t, t)$ be the smallest integer n such that every 3-coloring of the edges of K_n contains a monochromatic K_t. Prove that

   (a) $R(t, t, t) \le 27^t$, and

   (b) derive an exponential lower bound for the 3-color Ramsey number $R(t, t, t)$. That is, for each t, come up with an n_t, and a 3-coloring of the edges of K_{n_t} such that there is no monochromatic K_t, where n_t grows exponentially with t.

   Note: I can do this with $n_t = 3^{t/2}$ for every even t (which is sufficient), but it is OK if you prove this with a different n_t, as long as yours grows exponentially with t.

4. There are 100 cities in a country, called "1", "2", ..., "100". The government would like to connect them all with a network, and it costs $\max\{|i - j|, 4\}$ to lay a wire between cities "i" and "j". (There is a minimum cost of 4 per wire.) Another company comes along, offering to create microwave links between city pairs at a cost of 2 each, with the catch that their technology can only connect city pairs of the form ("i", "$2i$"). For example, it would cost 30 to lay a wire between cities 31 and 61, 4 to lay a wire between cities 31 and 32, and 2 to microwave-link 30 and 60. What is the minimum cost to create this connected network?