

# 21-228 Discrete Mathematics

## Assignment 3

Due Fri Feb 15, at start of class

**Notes:** Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation. Each question is worth 10 points.

1. In class, we proved Dirichlet's theorem, which states that for any real number  $\alpha$ , there are integers  $p$  and  $q$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

In other words, every real number has a pretty good rational approximation. What if we would like to approximate two different real numbers using the same denominator? Again, it's easy to see that for any real  $\alpha, \beta$ , there are integers  $p, p'$  and  $q$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q} \quad \text{and} \quad \left| \beta - \frac{p'}{q} \right| < \frac{1}{q}.$$

In fact, for any real  $\alpha, \beta$ , there are integers  $p, p'$ , and  $q$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{3/2}} \quad \text{and} \quad \left| \beta - \frac{p'}{q} \right| < \frac{1}{q^{3/2}}.$$

To prove this, show that for any real  $\alpha, \beta$  and any positive integer  $N$ , there are integers  $p$  and  $p'$ , and an integer  $q$  satisfying  $1 \leq q \leq N^2$ , such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{qN} \quad \text{and} \quad \left| \beta - \frac{p'}{q} \right| < \frac{1}{qN}.$$

**Extra credit (10 pts).** Use this to solve the following Putnam B6. For each positive real number  $\alpha$ , let  $S(\alpha)$  denote the set  $\{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}$ . Prove that  $\{1, 2, 3, \dots\}$  cannot be expressed as the disjoint union of three sets  $S(\alpha), S(\beta)$  and  $S(\gamma)$ .

2. How many rearrangements of the word "DOCUMENT" have the three vowels all next to each other? For example, "DOEUCMNT" counts, but not "DOCUEMNT".
3. A word over the alphabet  $\{a, b, c, \dots, z\}$  is called *increasing* if its letters appear in alphabetical order. For example, *boost* is increasing, but *hinder* is not. How many increasing words are there of length 52? Count non-English words, so that the answer is not zero. Answers may be expressed in terms of factorials or binomial coefficients, but summation notation and ellipses may not be used.
4. Prove that if we move straight down in Pascal's triangle (visiting every other row), then the numbers we see are increasing.