21-228 Discrete Mathematics

Assignment 1

Due Fri Jan 18, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Let $n \geq 3$. How many subsets of $\{1, 2, \ldots, n\}$ contain exactly two of the integers 1, 2, 3? For example, $\{1, 2, 7, 9\}$ and $\{1, 3, 9\}$ would count, but $\{1, 7, 9\}$ would not.

2. King Kong has escaped, and is at the southwest corner of Central Park (59th St / 8th Ave). He wants to get to the Empire State Building (34th St / 5th Ave) as quickly as possible, but he must avoid Times Square (42nd St / 7th Ave). If he always takes the most direct route, in how many ways can this be done? Assume the streets form a perfect grid, i.e., ignore Broadway, parks, etc. Answers may be expressed in terms of factorials or binomial coefficients, but summation notation and ellipses may not be used.

3. There are two cities named *Oddburg* and *Evenburg*. Each has $n \geq 2$ people. These cities have some strange laws regulating the clubs that can be formed by their citizens. (Oddburg and Evenburg are at war, so no club contains citizens from both at the same time.) The laws are:

   Oddburg: (1) no two clubs have exactly the same set of members, (2) every pair of clubs has an even number of members in common, and (3) every club has **odd** size.

   Evenburg: (1) no two clubs have exactly the same set of members, (2) every pair of clubs has an even number of members in common, and (3) every club has **even** size.

   Note that only the 3rd rules are different.

   Show that in one of the cities, it is possible to have an exponential number of different clubs. That is, show that there is some positive number c, e.g., $c = 0.001$, such that for any value of n, the number of possible clubs is at least 2^{cn}.

   (There is a beautiful theorem which shows that the other city only supports a maximum of n clubs, which is not exponential.)

4. Recall that $A \triangle B$ denotes the symmetric difference of A and B, i.e., the set of all x that belong to exactly one of A or B. Simplify:

$$(A \triangle B) \triangle (B \triangle C) \triangle (C \triangle A).$$

   If you wish, you may express your answer in the form of a Venn diagram, with the final set shaded in.