1 Problems

Putnam 1987/A1. Curves $A$, $B$, $C$, and $D$ are defined in the plane as follows:\footnote{The equations defining $A$ and $B$ are indeterminate at $(0, 0)$. The point $(0, 0)$ belongs to neither.}

\[ A = \left\{ (x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2} \right\}, \]
\[ B = \left\{ (x, y) : 2xy + \frac{y}{x^2 + y^2} = 3 \right\}, \]
\[ C = \left\{ (x, y) : x^3 - 3xy^2 + 3y = 1 \right\}, \]
\[ D = \left\{ (x, y) : 3x^2y - 3x - y^3 = 0 \right\}. \]

Prove that $A \cap B = C \cap D$.

Putnam 1987/A2. The sequence of digits

\[ 123456789101112131415161718192021 \ldots \]

is obtained by writing the positive integers in order. If the $10^n$-th digit in this sequence occurs in the part of the sequence in which the $m$-digit numbers are placed, define $f(n)$ to be $m$. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

Putnam 1987/A3. For all real $x$, the real-valued function $y = f(x)$ satisfies

\[ y'' - 2y' + y = 2e^x. \]

(a) If $f(x) > 0$ for all real $x$, must $f'(x) > 0$ for all real $x$? Explain.

(b) If $f'(x) > 0$ for all real $x$, must $f(x) > 0$ for all real $x$? Explain.