12. Integer polynomials

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1 Classical results

Vandermonde determinant. Let \( a_0, a_1, \ldots, a_n \) be distinct numbers, and let \( b_0, b_1, \ldots, b_n \) be arbitrary (possibly equal to each other or to any of the \( a_i \)). Then there is a unique polynomial \( p(x) = c_n x^n + \cdots + c_0 \) such that \( p(a_i) = b_i \) for all \( 0 \leq i \leq n \).

Lagrange interpolation. An expression for the above polynomial is

\[
p(x) = \sum_{i=0}^{n} \frac{b_i}{\prod_{j \neq i} (a_i - a_j)} \prod_{j \neq i} (x - a_j).
\]

Fermat’s Last Theorem. The equation \( x^n + y^n = z^n \) has no positive integer solutions \((x, y, z, n)\) with \( n \geq 3 \).

2 Problems

Putnam 1940/A1. Let \( p(x) \) be a polynomial with integer coefficients. Suppose that for some positive integer \( c \), none of \( p(1), p(2), \ldots, p(c) \) are divisible by \( c \). Prove that \( p(b) \) is not zero for any integer \( b \).

Putnam 1947/B5. Let \( p(x) \) be the polynomial \( (x - a)(x - b)(x - c)(x - d) \). Assume \( p(x) = 0 \) has four distinct integral roots and that \( p(x) = 4 \) has an integral root \( k \). Show that \( k \) is the mean of \( a, b, c, d \).

Putnam 1953/B2. Let \( p(x) \) be a real polynomial of degree \( n \) such that \( p(m) \) is integral for all integers \( m \). Show that if \( k \) is a coefficient of \( p(x) \), then \( n!k \) is an integer.

Putnam 1940/B5. Find all rational triples \((a, b, c)\) for which \( a, b, c \) are the roots of \( x^3 + ax^2 + bx + c = 0 \).

Putnam 1955/A6. For what positive integers \( n \) does the polynomial \( p(x) = x^n + (2 + x)^n + (2 - x)^n \) have a rational root?

Putnam 1950/A6. Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \), and suppose that each \( a_n \) is 0 or 1.

1. Show that if \( f(1/2) \) is rational, then \( f(x) \) has the form \( p(x)/q(x) \) for some integer polynomials \( p(x) \) and \( q(x) \).

2. Show that if \( f(1/2) \) is not rational, then \( f(x) \) does not have the form \( p(x)/q(x) \) for any integer polynomials \( p(x) \) and \( q(x) \).

3 Homework

Please write up solutions to two of the problems, to turn in at next week’s meeting. One of them may be a problem that we discussed in class.