11. Geometry

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1 Classical results

Archimedes' Principle. Let a sphere of radius 1 be inscribed inside a cylinder of radius 1 and height 2. Then for any two planes parallel to the base of the cylinder, the cylindrical strip that they cut out of the cylinder has surface area equal to that of the strip that they cut out of the sphere.

2 Problems

Putnam 1946/B1. Two circles C_1 and C_2 intersect at A and B. C_1 has radius 1. L denotes the arc AB of C_2 which lies inside C_1. L divides C_1 into two parts of equal area. Show L has length greater than 2.

Putnam 1955/B3. Let S be a spherical cap with distance taken along great circles. Show that we cannot find a distance preserving map from S to the plane.

Putnam 1958/B3. Show that if X is a square of side 1 and $X = A \cup B$, then A or B has diameter at least $\sqrt{5}/2$. Show that we can find A and B both having diameter at most $\sqrt{5}/2$. The diameter of a set is the maximum (strictly speaking, the supremum) of all distances between pairs of points in the set.

Putnam 1951/A2. Let k be a positive real and let $P_1, P_2, \ldots, P_n$ be points in the plane. What is the locus of P such that $\sum PP_i^2 = k$? State in geometric terms the conditions on k for such points P to exist.

Putnam 1957/A5. Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.

Putnam 1958/A7. Show that we cannot place 10 unit squares in the plane so that no two have an interior point in common and one has a point in common with each of the others.

Putnam 1956/A6. Find $f : \mathbb{R} \to \mathbb{R}$ which preserves all rational distances but not all distances. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves all rational distances then it preserves all distances.

Putnam 1992/A6. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)