5. Continuous functional equations

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1 Functional equations

Cauchy. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that there must be a real number c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Putnam 1959/A3, made continuous. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function which satisfies $f(z) + zf(1-z) = 1 + z$ for all $z \in \mathbb{C}$. Determine all possible such functions f. (Note: the original problem did not require f to be continuous. You can solve that one for fun if you would like.)

Putnam 1947/A2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies $f(\sqrt{x^2+y^2}) = f(x)f(y)$ for all real x and y. Show that $f(x) = f(1)^{x^2}$.

Putnam 1996/A6 (modern A6). Let $c > 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : R \to R$ such that $f(x) = f(x^2 + c)$ for all $x \in R$. Note that R denotes the set of real numbers.

2 Convergence

Putnam 1955/B6. Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be a function which satisfies $\lim_{n\to\infty} f(n) = 0$. Show that there are only finitely many solutions to the equation $f(x) + f(y) + f(z) = 1$.

Putnam 1954/A5. Let $f : (0,1) \to \mathbb{R}$ be a function which satisfies $\lim_{x\to 0} f(x) = 0$. Suppose that as $x \to 0$, we have $f(x) - f(x/2) = o(x)$, which means that $\lim_{x\to 0} \frac{f(x)-f(x/2)}{x} = 0$. Show that $f(x) = o(x)$ as $x \to 0$ (i.e., that $\lim_{x\to 0} \frac{f(x)}{x} = 0$).

Putnam 1951/A7. Let $a_1, a_2, \ldots$ be a sequence of real numbers for which the sum $\sum_{i=1}^{\infty} a_i$ converges. (This means that the sequence of partial sums $a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \ldots$ is itself a convergent sequence.) Show that the sum $\sum_{i=1}^{\infty} \frac{a_i}{i}$ also converges.