3. Polynomials

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1 Classical results

1. Find a nice expression for the derivative of the polynomial \((x - 1)(x - 2)(x - 3)^2\).

2. Let \(p(x) = a_nx^n + \cdots + a_0\) be a polynomial which satisfies \(p(-x) = p(x)\) for every real \(x\). Prove that \(a_i = 0\) for every odd \(i\).

2 Problems

Putnam 1958/A1. Show that the real polynomial \(\sum_0^n a_i x^i\) has at least one real root if \(\sum \frac{a_i}{i+1} = 0\).

Putnam 1959/A1. Prove that we can find a real polynomial \(p(y)\) such that \(p(x-1/x) = x^n - 1/x^n\) (where \(n\) is a positive integer) iff \(n\) is odd.

Putnam 1938/A3. The roots of \(x^3 + ax^2 + bx + c = 0\) are \(\alpha, \beta, \) and \(\gamma\). Find the cubic whose roots are \(\alpha^3, \beta^3,\) and \(\gamma^3\).

Putnam 1940/A6. Let \(p(x)\) be a polynomial with real coefficients, and let \(r(x)\) be the polynomial defined by the derivative \(r(x) = p'(x)\). Suppose that there are positive integers \(a\) and \(b\) for which \(r^a(x)\) divides \(p^b(x)\) as polynomials. Prove that for some real numbers \(A\) and \(\alpha\), and for some integer \(n\), we have \(p(x) = A(x - \alpha)^n\).

Putnam 1947/B4. \(p(z) = z^2 + az + b\) has complex coefficients. \(|p(z)| = 1\) on the unit circle \(|z| = 1\). Show that \(a = b = 0\).

Putnam 1956/B7. Let \(p(z)\) and \(q(z)\) be complex polynomials with the same set of roots (but possibly different multiplicities). Suppose that \(p(z) + 1\) and \(q(z) + 1\) also have the same set of roots. Show that \(p(z) = q(z)\).

Putnam 1957/A4. Let \(p(z)\) be a polynomial of degree \(n\) with complex coefficients. Its roots (in the complex plane) can be covered by a disk of radius \(r\). Show that for any complex \(k\), the roots of \(np(z) - kp'(z)\) can be covered by a disk of radius \(r + |k|\).

3 Homework

Please write up solutions to two of the problems, to turn in at next week’s meeting. One of them may be a problem that we discussed in class.