1 Problems

**Putnam 1991/B1.** For each integer \( n \geq 0 \), let \( S(n) = n - m^2 \), where \( m \) is the greatest integer with \( m^2 \leq n \). Define a sequence \((a_k)_{k=0}^{\infty}\) by \( a_0 = A \) and \( a_{k+1} = a_k + S(a_k) \) for \( k \geq 0 \). For what positive integers \( A \) is this sequence eventually constant?

**Putnam 1991/B2.** Suppose \( f \) and \( g \) are non-constant, differentiable, real-valued functions defined on \(( -\infty, \infty )\). Furthermore, suppose that for each pair of real numbers \( x \) and \( y \),

\[
\begin{align*}
    f(x + y) &= f(x)f(y) - g(x)g(y), \\
    g(x + y) &= f(x)g(y) + g(x)f(y).
\end{align*}
\]

If \( f'(0) = 0 \), prove that \( (f(x))^2 + (g(x))^2 = 1 \) for all \( x \).

**Putnam 1991/B3.** Does there exist a real number \( L \) such that, if \( m \) and \( n \) are integers greater than \( L \), then an \( m \times n \) rectangle may be expressed as a union of \( 4 \times 6 \) and \( 5 \times 7 \) rectangles, any two of which intersect at most along their boundaries?