10. Linear Algebra

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CMU Putnam Seminar, Fall 2011

1 Determinants (and singularity)

**Putnam 2008/A2.** Alan and Barbara take turns filling entries of an initially empty $2008 \times 2008$ array. Alan plays first. At each turn, a player places a real number in a vacant entry. When all entries are filled, Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

**VTRMC 2004/1.** Let $I$ denote the $2 \times 2$ identity matrix \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix};
\] let $A$, $B$, and $C$ be arbitrary $2 \times 2$ real matrices; finally, define the $4 \times 4$ matrices $M$ and $N$ by $M = \begin{bmatrix} I & A \\ B & C \end{bmatrix}$ and $N = \begin{bmatrix} I & B \\ A & C \end{bmatrix}$. Is it possible that exactly one of $M$ and $N$ is invertible?

**VTRMC 2007/6.** Let $A$ and $B$ be symmetric real $n \times n$ matrices. Suppose there are $n \times n$ matrices $X, Y$ such that $\det(AX + BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$.

2 Thinking in Matrices

**Well-known result.** A matrix $A$ is **symmetric** if $A^T = A$ and **skew-symmetric** if $A^T = -A$. Prove that any matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

**Putnam 1991/A2.** Let $A$ and $B$ be distinct $n \times n$ real matrices. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

**VTRMC 2010/1, modified.** Let $A$ be a $n \times n$ integer matrix. Suppose $I + A + A^2 + \cdots + A^{100} = 0$. Show that $A^k + A^{k+1} + \cdots + A^{100}$ has determinant $\pm 1$ for every positive integer $k \leq 100$.

3 Rank and dimension

**Putnam 2003/B1.** Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that, for all $x$ and $y$,

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)?$$

**VTRMC 2005/7.** Let $A$ be a $5 \times 10$ real matrix. Suppose every $5 \times 1$ real matrix (i.e. column vector in 5 dimensions) can be written in the form $Au$, where $u$ is a $10 \times 1$ real matrix. Prove that every $5 \times 1$ real matrix can be written in the form $AA^Tv$ where $v$ is a $5 \times 1$ real matrix.

**Putnam 2009/B4.** Say that a polynomial with real coefficients in two variables, $x, y$, is **balanced** if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space $V$ over $\mathbb{R}$. Find the dimension of $V$. 