Elementary methods

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1 Problems

Putnam 1995/A1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, not necessarily distinct, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T, and that the product of any three (not necessarily distinct) elements of U is in U, show that at least one of the two sets T, U is closed under multiplication.

Putnam 1997/B1. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For example, $\{1.7\} = 0.3$. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

Here, $\min(a, b)$ denotes the minimum of a and b.

Putnam 1995/B3. To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate

$$\det\begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50.$$

Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

Putnam 1995/B4. Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \dots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

Putnam 2000/A3. The octagon $ABCDEFGH$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $ACEG$ is a square of area 5, and the polygon $BDFH$ is a rectangle of area 4, find the maximum possible area of the octagon.