Combinatorics

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1 Problems

Putnam 2003/A1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: $4, 2+2, 1+1+2, 1+1+1+1$.

Putnam 2001/B1. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots (k-1)n + n.$$

Color the squares of the grid so that exactly half of the squares in each row and each column are red, and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Putnam 2002/A3. Let $n \ge 2$ be an integer and let T_n be the number of non-empty subsets S of $\{1, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Putnam 2006/A4. Let $S = \{1, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$. What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Easy. Let G be an n-vertex graph with all degrees equal to d. An *independent set* is a collection of vertices which are mutually nonadjacent. Show that G has an independent set of size at least $\frac{n}{d+1}$.

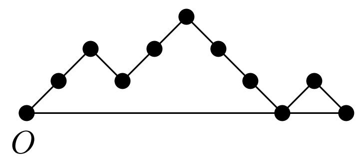
Turán. Suppose we only know that the average degree of G is d. Show that G still has an independent set of size at least $\frac{n}{d+1}$.

Turán. Suppose that G contains no clique of size t. What is the maximum number of edges that G can have?

Putnam 2005/B4. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, \ldots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$.

2 Bonus problems

Putnam 2003/A5. A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

Putnam 2004/A5. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares p and q are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.