Geometry

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1 Problems

Putnam 2008/B1. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

Classical (Art Gallery Problem). The floor plan of a single-floor art gallery can be considered as a (not necessarily convex) polygon with n vertices. Prove that it is always possible to position $\lfloor n/3 \rfloor$ stationary guards such that every point inside the gallery has a line-of-sight connection to some guard.

Classical. Prove that every (not necessarily convex) polygon has a triangulation.

Putnam 1998/A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side length of the cube?

GA 632. Let n be a positive integer. Prove that if the vertices of a $(2n+1)$-dimensional cube have integer coordinates, then the length of the edge of the cube is an integer.

Putnam 1999/B1. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$.

Putnam 1998/B3. Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers.

2 Bonus problems

Putnam 2004/B4. Let $n \geq 2$ be a positive integer, and let $\theta = \frac{2\pi}{n}$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point $(k, 0)$. For an arbitrary point $P = (x, y)$, find, and simplify, the result of applying, in order, R_1, R_2, $\ldots$, R_n to P. That is, compute $R_n(R_{n-1}(\cdots R_1(P) \cdots))$.

Putnam 2000/A3. The octagon $ABCDEFGH$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $ACEG$ is a square of area 5, and the polygon $BDFH$ is a rectangle of area 4, find the maximum possible area of the octagon.