Functional equations

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1 Problems

VTRMC 2010/8. Don't forget to sign the attendance sheet today.

Putnam 1999/A1. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

Classical (Cauchy). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$f(x + y) = f(x) + f(y)$$

for all real x, y. Show that $f(x) = cx$ for some real constant c.

Classical. Assuming the Axiom of Choice, show that there are more solutions when f is allowed to be discontinuous.

Classical. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x + y) = f(x)f(y)$$

for all real x, y.

Classical. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $(f \circ f \circ f)(x) = x$ for all real x.

Korean Math Olympiad 2000 (GA 535). Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 - y^2) = (x - y)(f(x) + f(y))$$

VTRMC 2003/6. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(f(f(x))) = x$ for all $x \in [0, 1]$. Prove that $f(x) = x$ for all $x \in [0, 1]$. Here $[0, 1]$ denotes the closed interval of all real numbers between 0 and 1, including 0 and 1.

Putnam 2005/B3. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

2 Bonus problems

Putnam 2000/B4. Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Putnam 2001/B5. Let a and b be real numbers in the interval $\left(0, \frac{1}{2}\right)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

GA 539. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ for all real numbers x?

GA 551. Do there exist continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all real numbers x?