Recursions

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1 Problems

VTRMC 2010/0. When and where is the VTRMC?

VTRMC 2008/2. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

VTRMC 2001/3. For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e. determine S_8).

Classical. How about the number of rectangles?

VTRMC 2004/3. A computer is programmed to randomly generate a string of six symbols using only the letters A, B, C. What is the probability that the string will not contain three consecutive A's?

VTRMC 2002/5. Let n be a positive integer. A bit string of length n is a sequence of n numbers consisting of 0's and 1's. Let $f(n)$ denote the number of bit strings of length n in which every 0 is surrounded by 1's. (Thus for $n = 5$, 11101 is allowed, but 10011 and 10110 are not allowed, and we have $f(3) = 2$, $f(4) = 3$.) Prove that $f(n) < (1.7)n$ for all n.

VTRMC 2005/3. We wish to tile a strip of n 1-inch by 1-inch squares. We can use dominos which are made up of two tiles which cover two adjacent squares, or 1-inch square tiles which cover one square. We may cover each square with one or two tiles and a tile can be above or below a domino on a square, but no part of a domino can be placed on any part of a different domino. We do not distinguish whether a domino is above or below a tile on a given square. Let $t(n)$ denote the number of ways the strip can be tiled according to the above rules. Thus for example, $t(1) = 2$ and $t(2) = 8$. Find a recurrence relation for $t(n)$, and use it to compute $t(6)$.

2 Bonus problems

Putnam 2005/A2. Let S be an $n \times 3$ chessboard, i.e., $S = \{(a, b) : a = 1, \ldots, n; b = 1, 2, 3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

(i) $p_i \in S$,

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \le i < 3n$,

(iii) for each $p \in S$ there is a unique i such that $p_i = p$.

How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

Putnam 2005/B4. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1,\ldots,x_n)$ of integers such that $|x_1|+\cdots+|x_n|\le m$. Show that $f(m,n)=f(n,m)$.

Putnam 2007/B3. Let $x_0=1$, and for $n\ge 0$, let

$$x_{n+1}=3x_n+\lfloor x_n\sqrt{5}\rfloor.$$

In particular, $x_1=5$, $x_2=26$, $x_3=136$, $x_4=712$. Find a closed-form expression for x_{2007}.