Number theory

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1 Problems

Gelca/Andreescu 733. Prove that there are infinitely many prime numbers of the form $4m - 1$, where m is an integer.

GA. The 9-digit number 2^{29} has 9 distinct digits. Which of the 10 possible digits 0–9 does not appear?

VTRMC 2009/2. Given that $40! = abc\ def\ 283\ 247\ 897\ 734\ 345\ 611\ 269\ 596\ 115\ 894\ 272\ pqr\ stu\ vwx$, find $p, q, r, s, t, u, v, w, x$, and then find a, b, c, d, e, f.

Putnam 2000/A2. Prove that there exist infinitely many integers n such that n, $n+1$, $n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

Putnam 2000/B2. Prove that the expression

$$\frac{\gcd(m, n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

GA 727. Let n, a, b be positive integers. Prove that $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$.

VTRMC 2006/3. Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2006)$ (e.g. the last digit of 2006 is 6).

GA. A positive integer is wrtten at each integer point in the plane ($\mathbb{Z}^2$), in such a way that each of these numbers is the arithmetic mean of its four neighbors. Prove that all of the numbers are equal.

2 Bonus problems

VTRMC 2009/6. Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square (i.e. of the form m^2 for some integer m).

Asia-Pacific Math Olympiad 1998 (from GA). Show that for any positive integers a and b, the product $(36a + b)(a + 36b)$ cannot be a power of 2.

Putnam 1939 (GA 711). Prove that no integer $n > 1$ divides $2^n - 1$.