Pigeonhole Principle

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14 September 2010

1 Problems

Classical. Given n integers, prove that some nonempty subset of them has sum divisible by n.

Paul Erdős. Let A be a set of $n+1$ integers from $\{1, \ldots, 2n\}$. Prove that some element of A divides another.

Putnam 2002/A2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Putnam 2006/B2. Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

Putnam 2000/B1. Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $\frac{4}{7}N$ values of j, $1 \leq j \leq N$.

VTRMC 2002/3. Let A and B be nonempty subsets of $S = \{1, 2, \ldots, 99\}$ (integers from 1 to 99 inclusive). Let a and b denote the number of elements in A and B respectively, and suppose $a + b = 100$. Prove that for each integer s in S, there are integers x in A and y in B such that $x + y$ is either s or $s + 99$.

Erdős-Szekeres. Prove that every sequence of n^2 distinct numbers contains a subsequence of length n which is monotone (i.e. either always increasing or always decreasing).

MOP 2007/7/1. A 100×100 array is filled with numbers from $\{1, \ldots, 100\}$, such that each number appears exactly 100 times. Prove that there is some row or column which contains at least 10 different numbers.

2 Bonus problems

Putnam 2006/A3. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

MOP 2004. A set S of numbers is called a *Sidon set* if it has the property that for every distinct $a, b, c, d \in S$, the sums $a + b$ and $c + d$ are distinct. (There are no repeated pairwise sums between elements of S.) A natural question is to ask how large a Sidon set can be, if, say, the numbers must be integers in $\{1, \ldots, 100\}$. Prove that there is no such Sidon set of size 16.

Putnam 1993/A4. Let $x_1, \ldots, x_{19}$ be positive integers less than or equal to 93. Let $y_1, \ldots, y_{93}$ be positive integers less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_i's.