1 Problems

Classical. Given $n$ integers, prove that some nonempty subset of them has sum divisible by $n$.

Paul Erdős. Let $A$ be a set of $n + 1$ integers from \{1, \ldots, 2n\}. Prove that some element of $A$ divides another.

Putnam 2002/A2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Putnam 2006/B2. Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of $n$ real numbers, there exists a non-empty subset $S$ of $X$ and an integer $m$ such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

Putnam 2000/B1. Let $a_j, b_j, c_j$ be integers for $1 \leq j \leq N$. Assume for each $j$, at least one of $a_j, b_j, c_j$ is odd. Show that there exist integers $r, s, t$ such that $ra_j + sb_j + tc_j$ is odd for at least $\frac{4}{7}N$ values of $j$, $1 \leq j \leq N$.

VTRMC 2002/3. Let $A$ and $B$ be nonempty subsets of $S = \{1, 2, \ldots, 99\}$ (integers from 1 to 99 inclusive). Let $a$ and $b$ denote the number of elements in $A$ and $B$ respectively, and suppose $a + b = 100$. Prove that for each integer $s$ in $S$, there are integers $x$ in $A$ and $y$ in $B$ such that $x + y$ is either $s$ or $s + 99$.

Erdős-Szekeres. Prove that every sequence of $n^2$ distinct numbers contains a subsequence of length $n$ which is monotone (i.e. either always increasing or always decreasing).

MOP 2007/7/1. A $100 \times 100$ array is filled with numbers from \{1, \ldots, 100\}, such that each number appears exactly 100 times. Prove that there is some row or column which contains at least 10 different numbers.

2 Bonus problems

Putnam 2006/A3. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k+2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

MOP 2004. A set $S$ of numbers is called a Sidon set if it has the property that for every distinct $a, b, c, d \in S$, the sums $a + b$ and $c + d$ are distinct. (There are no repeated pairwise sums between elements of $S$.) A natural question is to ask how large a Sidon set can be, if, say, the numbers must be integers in \{1, \ldots, 100\}. Prove that there is no such Sidon set of size 16.

Putnam 1993/A4. Let $x_1, \ldots, x_{19}$ be positive integers less than or equal to 93. Let $y_1, \ldots, y_{93}$ be positive integers less than or equal to 19. Prove that there exists a (nonempty) sum of some $x_i$’s equal to a sum of some $y_i$’s.